Thickness and Competition in Ride-sharing Markets

Afshin Nikzad*

October 2017

Abstract

We study the effects of thickness and competition on the equilibria of ride-sharing markets, in which price-setting firms provide platforms to match customers ("riders") and workers ("drivers"). To study thickness, we vary the number of potential workers ("the labor pool") and, to study competition, we change the number of firms from one to two. When the market is sufficiently thick, wage and workers' average welfare decrease with size of the labor pool. Otherwise, wage and workers' average welfare increase with the labor pool, reversing the prediction by the law of demand. Intuitively, workers are "complements" in a thin market –their wage and average welfare goes up with the labor pool– but they become "substitutes" and compete with each other in a thick market.

We demonstrate that a similar insight holds in another context: consider improving the matching technology, i.e. improving the matching algorithm of the firm so that service quality goes up, given the same labor supply. We show that improving the matching technology can be like increasing the labor pool, benefiting workers when the market is not sufficiently thick, while otherwise reducing their wage and average welfare. In other words, matching technology complements labor in a thin market, but substitutes it in a thick market.

We study competition by comparing the monopoly and duopoly equilibria. We find that competition benefits workers: their wage and average welfare are always higher in the duopoly equilibrium. However, the effect of competition on price and customers' average welfare depends on thickness, because firms compete for workers as well as for customers. When the market is not sufficiently thick, there is an adverse effect of competition on customers: price is higher and customers' average welfare is lower in the duopoly equilibrium.

*Department of Economics, Stanford University. Email: nikzad@stanford.edu

1 Introduction

Ride-sharing platforms (Uber and Lyft) and online labor platforms (Upwork and Taskrabbit) are examples of two-sided platforms which match service providers (workers) on one side with service requesters (customers) on the other side. Such platforms invest heavily in improving service quality for customers, often, by expanding their labor pool, as well as by improving their resource allocation strategies to provide better matches or recommendations. Understanding the impact of these activities on both sides of the market is a central concern to these platforms and also to the market regulators. We study these effects in the context of ride-sharing markets.

We study the effects of *thickness*, *improved matching technology*, and *competition* on the equilibria of ride-sharing markets, in which price-setting firms provide platforms to match customers ("riders") and workers ("drivers"). To study thickness, we vary the number of potential workers ("the labor pool"). By improved matching technology, we mean improving the matching algorithm used by the firm so that service quality goes up, given the same labor supply. Finally, by competition we mean entrance of a rival firm.

Varying the size of the labor pool leads to both expected and unexpected implications. By analogy to a classical marketplace, increasing the labor pool increases supply and makes it cheaper to provide any level of service, so customers' average welfare generally goes up. What is different about the ride-sharing market is that a larger labor pool can lead the firm to offer *higher* wages to workers, so that the workers' average welfare increases, too. The intuition for this reversal is founded in the observation that increasing size of the labor pool pressures the equilibrium wage in two ways. First, increasing the labor pool increases the number of workers at any wage, raising quality (for example, through shorter wait times for riders) and pushing up the firm's marginal expenditure to improve quality. Second, increasing the labor pool also reduces the firm's marginal expenditure to acquire an additional worker, pushing down the firm's marginal expenditure to improve quality. We show that this trade-off can go either way, and a larger labor pool can make it profitable for the firm to offer a higher wage.

We study this trade-off and its determinants in Section 4. We show that when the market is not sufficiently thick, workers' wage, their average welfare, and their average employment time *increase* as the labor pool increases (Theorem 4.3). So, workers are "complements" in a thin market, whereas they become "substitutes" and compete with each other in a thick market. (This suggests that in a thin market it can be easier for the firm to attract workers.) To see the intuition, recall the trade-off that we discussed above. When the market is not sufficiently thick, service quality is low and marginal increase in quality for an increase in wage is high, tilting the trade-off towards increasing wage.

Besides expanding the labor pool, platforms can also improve service quality by improving their resource allocation strategies. We study the effect of improved *matching technology* in ride-sharing markets, i.e. improving the matching algorithm of the firm so that service quality goes up given the same labor supply. We show that improving the matching technology can be like increasing the labor pool, benefiting workers when the market is not sufficiently thick, while otherwise reducing their wage and average welfare. In other words, matching technology complements labor in thin markets, and substitutes it in thick markets. Section 5 studies this effect and its determinants.

We study competition by comparing monopoly and duopoly and find that competition benefits workers: their average welfare and wages are always higher in the duopoly equilibrium. However, the effect of competition on the price and customers' average welfare depends on thickness: when the market is sufficiently thick, price is lower and customers' average welfare is higher in the duopoly equilibrium; but when the market is less thick, price is higher and customers' average welfare is lower.

The intuition is simple. There are two main forces affecting the duopoly price. Competition over customers pushes the customer price down, but there is another effect of competition that is adverse to customers: competition over workers raises the firms' costs, pushing the customer price up. The net effect of competition on price depends on the strength of these forces. When the market is thin, competition over workers dominates competition over customers, and the duopoly price for customers is higher than the monopoly price. We study this effect and its determinants in Section 6

Finally, we point out to some of the prominent features of ride-sharing markets which are detrimental to the effects that we introduce. A feature related to the effect of competition is that the multihoming side (workers, who can accept ride-requests from both firms) cannot drive for both firms at the same time. The significant consequence is that competition always benefits the multihoming side but has an adverse effect on the singlehoming side (customers) in thin markets, as shown by Theorem 6.6 and Theorem 6.7. This is in contrast to the results in the classic two-sided platforms literature, where the singlehoming side typically benefits from competition, and the multihoming side could have all its surplus extracted, as firms do not directly compete for them [Armstrong 2005, Lam 2017]. Thereby, fundamental equilibrium properties in two-sided platforms can depend on the marketplace design details.

Another feature of ride-sharing markets that plays a crucial role in the effects of thickness and improved technology is convexity of the waiting time of a customer in the number of available idle workers: the marginal reduction in waiting time for an additional idle worker is decreasing in the number of idle workers. This and some of our other assumptions are discussed in Section 7. Similar convexities could be present in other two-sided marketplaces as well, deriving effects similar to the ones that we introduce here. However, ride-sharing markets are the only marketplace where we have established the existence of these effects at this level of generality. This highlights the prominent role of waiting times and their convexity in the number of idle workers in ride-sharing.

2 Related work

In this section we review some of the relevant work in the literature covering two-sided platforms, the economics of networks and the network effects, and dynamic pricing.

The literature on platform competition and two-sided markets offers generic insights on how the market equilibria change under different governance structures (e.g. a social welfare or a profit maximizing planner) and highlights the significance of some of the key structural components of two-sided markets, such as user heterogeneity and multi-homing and single-homing of users, among others. Some of these works are reviewed below. They consider different comparative statics than we do in this study. Also, they are generally tailored for different market structures, such as credit card markets. Such differences in design details can create significantly different equilibrium properties, as we briefly explained in Section 1.

[Rochet and Tirole 2003, Tirole and Rochet 2006] introduce a generic framework and use it to compare the end-user surpluses for different planners, study the determinants of the business model (the favorability of the price structure on each side of the market), and investigate different membership structures.

[Weyl 2010] highlights the role of user heterogeneity in normative properties and comparative statics of two-sided markets. He provides a reformulation of a platform's problem that allows for user heterogeneity in income or scale, and is in terms of the allocation choice, rather than prices. [White and Weyl 2010] suggest Insulated Equilibrium as a novel equilibrium notion, and show that under this notion, the impact of competition (defined by the level of product differentiation) on efficiency depends on heterogeneity in users' valuations for network effects.

[Caillaud and Jullien 2003] and [Armstrong 2005] elaborate the role of single-homing and multi-homing users on the market equilibria and users' surplus, and show that the single-homing side is treated favorably. [Armstrong and Wright 2007] argue that multi-homing users could result in "competitive bottlenecks" in a market and study exclusive deals for preventing multi-homing. More precisely, they show that when platforms are viewed as homogenous by multi-homing users but heterogeneous by single-homing users, they do not compete directly for multi-homing users, and instead, choose to compete for them indirectly by subsidizing single-homing users to join. In contrast, in this paper we find competition to be beneficial to the multi-homing side (workers), while being beneficial to the single-homing side only when the labor pool is sufficiently large.

There is also extensive literature on network externalities and economics of networks; [Shy 2011] gives a brief survey. One of the generic intuitions is that expanding both sides of a platform simultaneously could benefit both sides (compared to the effect of thickness in this work, where expanding the labor side alone is beneficial to the same side in thin markets). Several work in this area study competition between firms in the presence of network externalities. [Katz and Shapiro 1985] show that firms' joint incentives for product compatibility are lower than the social incentives. [Economides 1996] explains that the existence of network externalities cannot be claimed as a reason in favor of a monopoly market structure, as their presence "does not reverse the standard welfare comparison between monopoly and competition".

[Cournot et al. 1927] show that non-integrated dual monopolists can quote higher prices than a single vertically integrated monopolist in the pricing of two perfectly complementary goods. The intuition is that dual monopolists face a less elastic demand and quote higher prices than a single vertically integrated monopolist. (An effect also known by "double-marginalization".) [Economides 1999] extends this result by showing that the product quality also will be higher under a single integrated monopolist when the quality choices are endogenous and the goods are perfect complements. We remark that the adverse effect of competition on customers that we introduce arises not because of the complementarity of the goods (in fact, the two goods provided by the two firms are substitutes), but because of the competition over the multihoming side.

There is also literature that explains the benefits of dynamic or surge pricing in ride-sharing markets. [Banerjee et al. 2015] conclude that although dynamic pricing does not necessarily yield higher performance than static pricing, it provides the benefits of optimal static pricing even under imperfect information about the system parameters. [Castillo et al. 2017] show that dynamic pricing could be used to avoid inefficient equilibria with long pickup times. Such equilibria could exist when the firm uses a greedy matching policy. These papers, however, set aside the long-run effects of thickness or competition on the platform's strategy. This paper is the first that we know of to study these interdependent effects.

3 Setup

The model is a dynamic steady-state model. A single firm intermediates between customers and workers. Potential customers have an arrival flow, the rate of which is taken to be constant over time, and equal to 1. There is also a mass m of workers, which we call the *labor pool*. The firm makes matches between customers and workers. Immediately upon arrival, a customer either requests service or departs the market. When a customer requests service, the firm matches her to a worker in the set of idle workers who accept the firm's wage offer. The firm selects the worker uniformly at random. When the firm makes a match between a customer and a worker, the worker serves the customer for a unit of time, and after that, the worker returns to the pool of workers where she waits for a new match and the customer departs the market. The firm posts a price p and a wage w, which are the payment from a customer to the firm and the payment from the firm to a worker, upon a match.

Next, we discuss the decision problems of the workers, the customers, and the firm.

Workers

Each worker has an outside option $r \sim F$, which represents the *opportunity cost* of the worker: the worker earns r per unit of time whenever she is not serving a customer.[1] The workers therefore have a simple decision problem: a worker with outside option r accepts service requests from the firm iff $r \leq w$. A worker who would decide to accept requests from the firm is called a *viable* worker. The total mass of viable workers is therefore equal to $mF(w)$, which we typically denote by λ, when m, w, F are clearly known from the context. We sometimes refer to a viable worker as a worker who has *joined* the firm.

The set of viable workers is partitioned into two subsets, namely *busy* and *idle* workers. The busy workers are the workers who are serving customers presently. The idle workers are the rest of the viable workers. We denote the mass of busy and idle workers with b, i, respectively.

[1] Our main finding also holds under the assumption that the worker earns the outside option r only when she does not join the firm, i.e. when she never accepts ride requests from the firm.

Customers

Each customer has a valuation $v \sim G$ for the service. A customer requests service (i.e. *joins* the firm) iff $v > p + c(i)$, where p is the price posted by the firm, and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a cost function where $c(i)$ denotes the cost incurred by the customer given than there are i idle workers available in the pool upon the customer's request. For example, one may consider $c(i)$ as (the monetary equivalent of) the customer's waiting cost. We assume that c is a decreasing function. *Payoff* of the customer from joining the firm is equal to $v - p - c(i)$.

For any fixed p, w offered by the firm, we can determine the rate of customers who join the firm, namely k. To this end, we write the *market clearing condition* according to which the rate of rides supplied is equal to the rate of rides demanded

$$k = 1 - G(p + c(mF(w) - k)). \tag{3.1}$$

On the right-hand side we have the rate of customers who join the firm, i.e. $\mathbb{P}_{v\sim G}\left[v > p + c(i)\right]$. Note that the left-hand side is strictly increasing in k, while the right-hand side is decreasing in k (holding all other variables fixed). This implies that there is a unique k satisfying the above equation. We typically write this k as a function of p, w and denote it by $k(p, w)$.

We use $k(p, w), \lambda(p, w)$ respectively to denote the rate of customers who join the firm and the mass of viable workers under price p and wage w. When p, w are clearly known from the context, we just use the notation k, λ.

Firm

The firm posts price and wage in order to maximize its "objective function". The main objective that we consider for the firm is profit-maximization.

Profit maximizing firm Under a fixed choice of p, w, the firm's profit function is defined by $\Pi(p, w) \equiv (p - w) \cdot k(p, w)$. The firm's objective is maximizing profit by choosing price and wage, i.e. the firm's problem is defined by

$$\begin{aligned} &\max_{p,w\geq 0} \Pi(p, w) \\ &\text{s.t. } \; k(p, w) \leq mF(w). \end{aligned} \tag{3.2}$$

(3.2) is a capacity constraint that states the mass of viable workers has to be at least as much as the rate of customers who join. The optimal solution to the firm's problem is called the *monopoly equilibrium*. We say a monopoly equilibrium *exists at m* when the firm serves a positive rate of customers at its optimal solution when given a labor pool of size m. (We emphasize that this definition excludes optimal solutions at which the firm's profit is equal to zero.) We say that an equilibrium is *non-binding* if its capacity constraint does not bind, i.e. when the number of idle workers is positive.

We also extend some of our results and observations to when the firm's objective is welfare maximization, e.g. maximizing some combination of customers' and workers' welfares. Unless otherwise stated, the firm is considered to be a profit maximizer.

Assumptions

Suppose that $F, G : [0,1] \to [0,1]$ are strictly increasing CDFs and are of the class $\mathbf{C^4}$.[2] Furthermore, through out the paper, we assume that F has a decreasing PDF. The interpretation is that there are fewer workers with higher outside options. (Assumptions and extensions are discussed in Section 7)

We assume that the function $c : [0,\infty) \to [0,\infty]$ is of the class $\mathbf{C^4}$, decreasing, strictly convex, and that $c(0) = 1$. We call such a function a *standard* cost function.

Observe that under the assumption $c(0) = 1$, no customers will join the firm if there are no idle workers available, because the support of G is the unit interval. Therefore, this assumption ensures that all monopoly equilibria are non-binding. Convexity of c has a simple interpretation: each additional idle worker decreases the waiting cost less than the previous one.[3] The smoothness assumptions on c, F, G could be relaxed, as some of our proofs require lower differentiability classes; the details about such relaxations are omitted.

Existence and uniqueness of the equilibrium A monopoly equilibrium exists when m is not "too small" (Theorem 4.3), and is *generically* unique, in the following sense. The profit function, $\Pi(p, w)$, is bounded and continuous (Lemma B.7). This guarantees the existence of at least one local maximum. Informally speaking, there are generically no two local maxima that give the same profit, which means that the global maximum is generically unique.[4] Throughout the entire draft, whenever we refer to a monopoly equilibrium in a formal statement, its existence and uniqueness are guaranteed, and the proof of it is included in the

[2] Recall that a function is of the the class $\mathbf{C^n}$ if its first n derivative exist and are continuous.

[3] Convexity of the cost function means that having one more idle worker is much more effective in decreasing the waiting cost when there are 10 idle workers available, than when there are 1000 idle workers available.

[4] When G is a regular distribution, a stronger property holds: uniqueness of the local maximum.

proof of that statement.

We use $p(m), w(m), k(m)$ to denote the equilibrium levels of price, wage, and customers' entrance rate as a function of m. When m is clearly known from the context, we sometimes use the notations p^*, w^*, k^*, respectively.

4 The effect of thickness

We begin this section with two examples about the effect of thickness on the equilibrium wage. After discussing the intuition behind the observed non-monotonicity in wage, we do a simple mathematical exercise in Subsection 4.2 to formalize the intuition. This simple exercise also explains the crucial role of the convexity of the cost function in the non-monotonicity of equilibrium wage. Finally, we extend the non-monotonicity observed in the example by showing that the equilibrium wage, workers' average welfare, and their average employment time increase with labor pool when the labor pool is not sufficiently large. (Subsection 4.3)

4.1 Examples

Monopolist

In this example, we plot the wage offered by a monopolist when the cost function c is an exponential cost function, i.e. $c(i) = e^{-\gamma i}$. We observe that the equilibrium wage increases with size of the labor pool when size of the labor pool is smaller than a certain threshold.

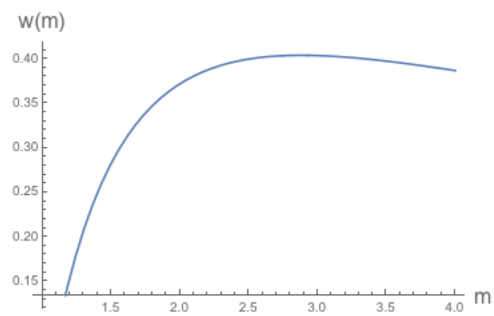


Figure 1: Equilibrium wage as a function of size of the labor pool (m); $\gamma = 1$ and F, G are the uniform distribution.

What is the intuition? As the size of the labor pool increases, the equilibrium wage is pressured by two forces. The first force works in favor of increasing wage: when m goes up, more workers join the firm for the same increase in wage.[5] This reduces the firm's marginal expenditure to acquire labor, which in turn pushes down the firm's marginal expenditure to improve service quality. This creates a force in favor of increasing the wage, which we call the *upward force*.

The second force is the *downward force*: as m goes up, the number of idle workers goes up as well (holding all else fixed).[6] Therefore, an additional worker decreases the waiting cost less than when size of the labor pool was smaller (because of the convexity of the cost function). This force pushes up the firm's marginal expenditure to improve service quality, and thereby creates a force in favor of decreasing the wage. Whether the wage goes up or down depends on which force is stronger.

We pin down the mathematical expressions corresponding to the upward and downward forces in a simple exercise in Subsection 4.2. We will see that when m is "large", waiting costs are low and additional idle workers do not decrease waiting costs much; the downward force is strong and therefore the equilibrium wage decreases with the labor pool. (Subsection 4.3)

Social planner

In this section, we repeat the same exercise that derives the law of demand, but for the case of ride-sharing markets. Figure 2 presents the typical proof-by-picture for the law of demand. The equilibrium level of wage is determined at the intersection of the inverse demand and supply functions, and it falls down as the labor supply goes up.

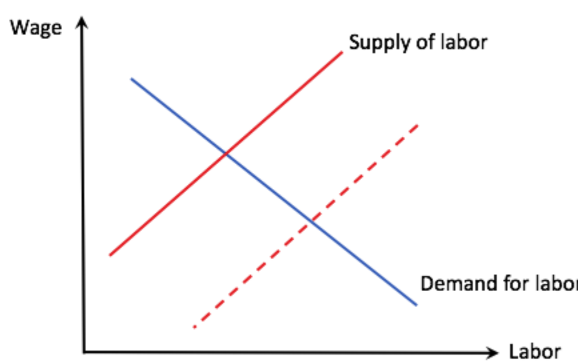


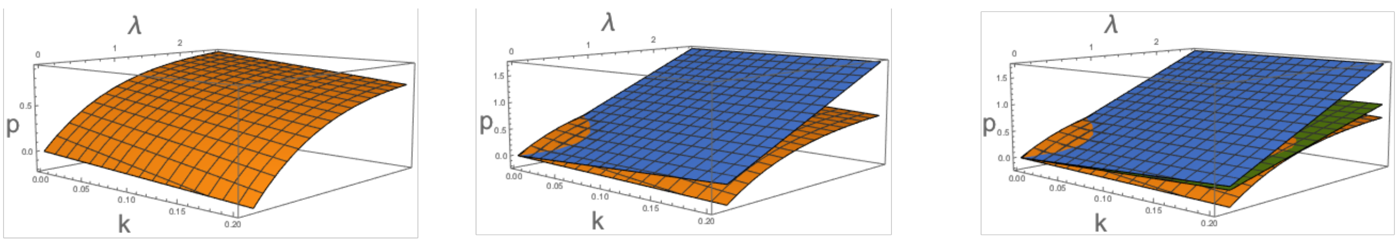
Figure 2: The law of demand

[5]To give a simple example, suppose that in a pool of 1000 workers, increasing wage by 1 cent convinces 10 more workers to join the firm. In a pool of 2000 workers, the same increase of 1 cent would convince, e.g., 20 more workers to join.

[6]This is easy to observe when k is held fixed. Even when we allow k to change endogenously as m changes (while holding price and wage fixed), the number of idle workers *always* go up for a marginal increase in m.

In Figure 3, we do the same exercise, but tailored for our model. Here, the inverse demand and inverse supply functions take two arguments as their input: the mass of viable drivers (λ) and the rate of customers who join (k). For expositional simplicity, we have supposed that the firm is a social planner and always sets price equal to wage. The exact objective function of the firm is defined later.

Concavity of the inverse demand function is a direct consequence of the convexity of the cost function c (see Figure 3(a)). The intersection of the inverse demand and inverse supply functions gives a continuum of equilibria (Figure 3(b)). To be able to proceed with our comparative statics, we choose a *selection rule* that selects one of these equilibria. For example, we consider the selection rule that chooses the equilibrium that serves the highest rate of customers. (The point corresponding to this equilibrium is observable in Figure 3(b)) As labor pool goes up, the inverse supply function shifts down. (Figure 3(c)) It can be observed that the level of wage at the equilibrium chosen by our selection rule goes up. The same exercise could be repeated for other planners as well.



(a) Inverse demand function

(b) The inverse demand and inverse supply functions

(c) The added surface (green color) corresponds to the inverse supply function at the higher m.

Figure 3

4.2 A mathematical exercise for capturing the upward and downward forces

We start this exercise by presenting an intuitive sufficient condition, and also a necessary and sufficient condition for the wage to go up with labor pool. Some of our results also hold when the firm is a social planner. In such cases, we refer to the related sections in the appendix. For expositional simplicity, we first suppose that F, G are the uniform distribution over the unit interval. After that, we relax the distributional assumptions and observe that the same identical sufficient condition still holds.

After defining some notation, we state the two conditions and the intuition behind them. With slight abuse of notation, define the cost function $c(mF(w)-k)$ as a function of three variables, m, w, k as follows:

$$c(m,w,k) = c(mF(w)-k).$$

This definition is used in defining the cross-partial

$$c_{m,w} \equiv \frac{\partial c(m,w,k)}{\partial m \partial w}.$$

Note that $c_{m,w}$ takes arguments m, w, k as its input. This cross-partial is the key part of the two conditions that we will present.

Recall that we use variables $p(m), w(m), k(m)$ to denote the equilibrium values as a function of m. When m is clearly known from the context, we sometimes use p^*, w^*, k^*, instead. By the usual convention, $p'(m), w'(m), k'(m)$ denote the derivatives of the equilibrium values with respect to m. When such derivatives are used in a formal statement, their existence is guaranteed by convention, and the proof of existence is included in the proof of that statement.

We are now ready to state the promised sufficient condition.

Proposition 4.1 (Sufficient condition)**.** *Let F, G be the uniform distribution. Then, there exists m_0 such that a monopoly equilibrium exists iff $m > m_0$. Moreover, the equilibrium is unique, and*

$$c_{m,w}(m, w^*, k^*) < 0 \Rightarrow w'(m) > 0.$$

We remark that the same identical condition remains sufficient if the planner is a social planner. Before we discuss the intuition, we give an interpretation for the cross-partial in simple words. The partial c_w could be interpreted as minus *marginal revenue per ride of increasing wage*: its output is a negative quantity that gives the reduction in the waiting cost for a marginal increase in wage, which could be charged to customers, and therefore is called the marginal revenue per ride.[7] The condition $c_{m,w} < 0$ states that the marginal revenue per ride of increasing wage goes up with m.

Intuition We provide an intuition based on the notions of the marginal cost and marginal revenue per ride of an increase in wage. (We remark that the same identical condition given

[7]Note that we are distinguishing between the terms revenue and profit. By the usual convention, revenue does not include service costs, but profit does.

in Proposition 4.1 remains sufficient if the planner is a social planner, where the mentioned marginal revenue and marginal cost functions would not be relevant.) The marginal cost per ride of an increase in wage is 1 (holding k fixed). Denote this quantity by $\text{MC} \equiv 1$. On the other hand, the marginal revenue per ride of an increase in wage is $-c_w(m, w, k)$: when wage goes up, waiting times go down, and therefore the monopolist can charge customers a higher price without changing their payoffs; the marginal amount that can be charged by the monopolist is $-c_w(m, w, k)$.[8] Denote this quantity by MR. At the monopoly equilibrium, we always have MR = MC, because otherwise, the monopolist can increase its profit by changing both price and wage by a small amount ϵ. (This is formally proved in Subsection B.1) To understand how a change in size of the labor pool, m, affects equilibrium wage, we should understand how a change in m affects MR, MC. The idea is that if $\frac{\partial \text{MR}}{\partial m} > \frac{\partial \text{MC}}{\partial m} = 0$, then the equilibrium wage increases with m. (See Figure 4 for an example) In simpler words, this condition says that a marginal increase in m has a positive impact on the marginal *profit* per ride of increasing wage. In what follows next, we make this intuition precise.

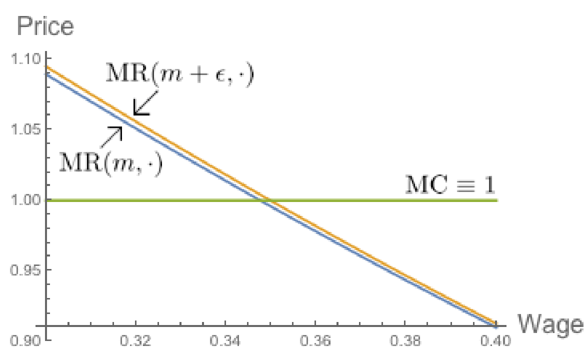


Figure 4: Plotting MR and MC as functions of w while increasing m by $\epsilon = 0.02$ and holding k fixed at its equilibrium value at m. $m = 1.8$ and c is the exponential cost function $c(i) = e^{-i}$. The intersection of MC and MR$(m, \cdot)$ gives the equilibrium wage at $m = 1.8$.

First, observe that $\frac{\partial \text{MC}}{\partial m} = 0$. To compute $\frac{\partial \text{MR}}{\partial m}$, we write MR as a function of three variables in the natural way, and then compute the partial as follows:

[8] In other words, this quantity is just the negative of the marginal rate of substitution of wage for price on any iso-quant of the customer's payoff function, $u(v) = v - p - c(i)$ for a customer with valuation v.

$$
\begin{aligned}
\mathrm{MR}(m,w,k) \equiv\ & m \cdot \big(-c'(mw-k) \big), \\
\frac{\partial \mathrm{MR}(m,w,k)}{\partial m} =\ & -c'(mw-k) \ + \ -mwc''(mw-k) \\
=\ & \underbrace{-c'(i)}_{\text{The upward force}} \ + \ \underbrace{-\lambda c''(i)}_{\text{The downward force}}
\end{aligned}
$$

Therefore, $\frac{\partial \mathrm{MR}}{\partial m} > \frac{\partial \mathrm{MC}}{\partial m}$ holds iff $\lambda c''(i) + c'(i) < 0$. But the latter condition is just the sufficient condition given in Proposition 4.1, because

$$
\begin{aligned}
c_{m,w}(m,w,k) &= \partial \frac{mc'(mw-k)}{\partial m} \\
&= c'(i) + mwc''(i) \\
&= c'(i) + \lambda c''(i).
\end{aligned}
$$

Now we can pin down the upward and downward forces that were discussed briefly earlier in Subsection 4.1; they correspond to the two terms in $-c'(i) - \lambda c''(i)$. The first term, $-c'(i)$, corresponds to the upward force: as m goes up, more workers will join the firm for the same increase in wage. The term $-c'(i)$ captures this effect as it is the coefficient of m in $\mathrm{MR} = m \cdot (-c'(i))$. Note that this term has a positive sign and works in favor of satisfying the sufficient condition (unlike the second term which has a negative sign). The second term, $-\lambda c''(i)$, corresponds to the downward force: as m goes up, the number of idle workers also goes up. Therefore, an additional worker decreases the waiting cost less than when m was smaller, by the convexity of $c(i)$.

Convexity of the cost function c has a crucial role in this mathematical exercise and in non-monotonicity of the equilibrium wage. When c is concave or affine, the equation $\mathrm{MR} = \mathrm{MC}$ does not hold anymore, because the monopoly solution would not be an interior solution of the firm's profit maximization problem, as we discuss in Section 7. There, we see that the monopolist employs just enough workers to provide the lowest possible waiting time for customers, which would imply that the equilibrium wage decreases with the labor pool. Section 7 also discusses the convexity assumption as an inherent characteristic of ride-sharing markets.

We finish the discussion of this exercise with two remarks. First, we can repeat the same exercise even when the uniformity assumptions on F, G are dismissed; Subsection B.3 shows that a similar sufficient condition holds for general distributions. Second, the necessary and sufficient condition is quite similar to the sufficient condition. The difference is that the right-hand side (0) is replaced with a positive term, which captures the effect of changes in k on the equilibrium wage. These changes were ignored in the sufficient condition.

Proposition 4.2 (Necessary and sufficient condition)**.** *Let F, G be the uniform distribution. Then,*

$$c_{m,w}(m, w^*, k^*) < \frac{k'(m)}{m(w^* - k'(m))} \Leftrightarrow w'(m) > 0.$$

4.3 The welfare effects of thickness

Here, we will show the equilibrium wage, workers' average welfare, and their average employment time all increase with the labor pool when the labor pool is not too large.

First, we formally define the notions of average employment time and workers' average welfare, which will be used in the theorem. Generally, we use $x(m)$ to denote the equilibrium value of a parameter x as a function of m, e.g., $w(m)$ denotes the equilibrium wage. Let the average employment time of workers be defined as $e(m) \equiv \frac{k(m)}{\lambda(m)}$. Also, define the workers' average welfare as their per round average earnings from wage and outside options: [9]

$$u^W(m) \equiv \frac{1}{F(w(m))} \cdot \int_0^{w(m)} (w(m) \cdot e(m) + r \cdot (1 - e(m)) \cdot F'(r)\,\mathrm{d}\,r. \tag{4.1}$$

For example, when F is the uniform distribution, the above expression simplifies to

$$u^W(m) = \frac{1}{2} \cdot \left(w(m) + \frac{k(m)}{m} \right),$$

which has a simple interpretation: the average worker always earns her outside option $r = \frac{w(m)}{2}$. In a fraction $e(m)$ of the time when she is serving a customer, she also earns an additional amount of $w(m) - r$. The uniformity assumption is not made in the theorem that follows.

Theorem 4.3. *There exists $\underline{m}$ such that a monopoly equilibrium exists at m iff $m > \underline{m}$, and there exists $\hat{m} > \underline{m}$ such that for all $m \in (\underline{m}, \hat{m})$, $w'(m), e'(m)$, and $(u^W)'(m)$ are positive.*

In words, Theorem 4.3 says that when the labor pool is not too large, workers do not compete with each other, and increasing the labor pool increases their average welfare, wage, and average employment time. However, when the labor pool becomes sufficiently

[9]The theorem that we will present also holds for some other welfare-related notions, such as per round average earnings from wage.

large, workers compete with each other, and all these parameters decrease with the labor pool. Intuitively, workers are *complements* in a thin market, but they become *substitutes* in a thick market.

One might also be interested in considering other measures of welfare. A possible example is a measure that excludes the earnings from outside options in (4.1) and only considers the earnings from wage. The above result holds for this measure as well.

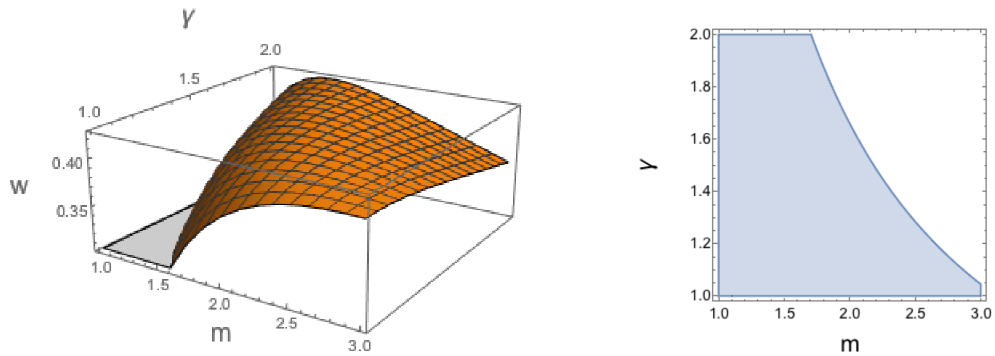
5 Matching technology

Aside from increasing the labor pool, platforms could improve their service quality by improving their resource allocation strategies. We study the effect of improved *matching technology*, i.e. improving the matching algorithm of the firm so that service quality goes up (waiting times go down), given the same labor supply. First, we demonstrate the effect of improved technology in an example, where we observe that improving the technology increases the wage and workers' average welfare when the labor pool is not large, and decreases these parameters when the pool becomes sufficiently large. In other words, matching technology complements labor in thin markets, and substitutes labor in thick markets. We present the intuition for this effect right after the example, and then set up a general model for improving matching technology and extend the example to a theorem.

5.1 Example

Let F, G be the uniform distribution over the unit interval. Also, let $c(x) = e^{-\gamma x}$. We are interested in the effect of increasing γ (i.e. improving the matching technology) on the equilibrium level of wage. In Figure 5(a), we plot equilibrium level of wage while varying m and γ. The shaded area in this figure is where the derivative of equilibrium wage with respect to γ is positive. Observe that for any fixed γ, there exists a threshold $\hat{m}_\gamma$ such that the equilibrium wage increases with γ iff $m < \hat{m}_\gamma$ (i.e., matching technology complements labor when the market is not sufficiently thick). An interesting observation is that the threshold $\hat{m}_\gamma$ decreases in γ. A simple explanation is given by interpreting improving technology as another way of "thickening the market": as the market becomes "thicker", improving technology becomes less favorable to workers.

Before extending the above example to more general cost functions and distributions, we discuss the intuition. To this end, consider the following thought experiment: Fix a monopoly equilibrium, and let p^*, w^*, k^*, i^* denote the equilibrium parameters, defined as



(a) Equilibrium wage.

(b) Wage increases with γ in the shaded area.

Figure 5

usual. Hold price and wage fixed, and then update the matching technology: this would correspond to changing the cost function c to a new cost function d such that $d(i) < c(i)$ holds for all positive i. Under the new matching technology, the number of customers who join the firm and the number of idle workers would change. Let $\tilde{i}$ and $\tilde{k}$ respectively denote the new parameters. The core of this thought experiment is based on the following fact: under the new matching technology, the equilibrium wage increases iff $c'(i^*) > d'(\tilde{i})$. In words, the equilibrium wage increases iff adding one more idle worker decreases the waiting time more under the new technology than under the old technology. To understand whether this would hold, we should look at the two effects involved: improving technology pressures the equilibrium wage in two ways, discussed below.

First, when technology is improved, service quality goes up, and therefore, the customers' demand for rides goes up. This also implies that the number of idle workers goes down, i.e. $\tilde{i} < i^*$. This creates a force that pushes down the firm's marginal expenditure to improve service quality (because, all else being equal, improving service quality is cheaper when the number of idle workers is smaller), which works in favor of increasing the equilibrium wage.

Second, at some levels of idle workers, namely i, an additional idle worker may decrease the waiting cost less under the new matching technology than under the old matching technology, i.e. $c'(i) < d'(i)$ may hold for some i. Note that both $c'(i), d'(i)$ are negative numbers, each saying how much waiting cost would go down for an additional idle worker. Although the function d lies below c, its derivative may be larger than the derivative of c at some i. When this inequality holds, it creates a force that pushes up the firm's marginal expenditure to improve service quality under the new technology, which works in favor of decreasing the equilibrium wage.

5.2 The welfare effects of improved matching technology

We set up a general model of matching technology by allowing the cost function to depend on the level of technology, which we denote by $\gamma \in (0, \infty)$. A higher value of γ corresponds to a "better" matching technology, as we will formalize soon. To this end, let the function $c : \mathbb{R}_+^2 \to \mathbb{R}_+$ denote the cost function, with $c(\gamma, i)$ being the customer's waiting cost when there are i idle workers available at level γ of technology.

We make the following assumptions on c: (i) c is *smooth*: c is continuous and its partials with respect to its arguments exist and are continuous. (ii) for any γ, the function $c(\gamma, \cdot)$ is a standard cost function, and (iii) cost goes down with technology: for any $\gamma_1 < \gamma_2$ and any positive i, $c(\gamma_1, i) > c(\gamma_2, i)$.

We need some notation to present the theorem. Let the variables $p(m,\gamma), w(m,\gamma), k(m,\gamma)$ denote the equilibrium values of these parameters as functions of m, γ. Also, define $e(m,\gamma) \equiv \frac{k(m,\gamma)}{\lambda(m,\gamma)}$. For a function $x(m,\gamma)$, we use the notation $x_i(m,\gamma)$ to denote the partial of $x(m,\gamma)$ with respect to its i-th argument, for $i \in \{1, 2\}$.

The following theorem shows that workers' average welfare, their wage, and their average employment time increase with technology when the market is not too thick.

Theorem 5.1. *For any $\gamma > 0$, there exists $\underline{m}_\gamma$ such that a monopoly equilibrium exists iff $m > \underline{m}_\gamma$, and there exists $\hat{m}_\gamma > \underline{m}_\gamma$ such that for all $m \in (\underline{m}_\gamma, \hat{m}_\gamma)$, $w_2(m,\gamma), e_2(m,\gamma)$, and $(u^W)_2(m,\gamma)$ are positive.*

6 The effect of competition

We study the effect of competition by comparing monopoly and duopoly equilibria. We start by setting up the duopoly model in Subsection 6.1, and then we present the results in Subsection 6.2. We find that workers' wage and average welfare are always higher in the duopoly equilibrium. However, the effect of competition on customers depends on the thickness: when the market is sufficiently thick, the price is lower and the customers' average welfare is higher in the duopoly equilibrium; but when the market is less thick, the price is higher and the customers' average welfare is lower.

There is a simple explanation. There are two main forces affecting the duopoly price. Competition over customers pushes the customer price down, while competition over workers raises the firms' costs, pushing the customer price up. The net effect of competition on price depends on the strength of these forces. When the market is thin, competition over workers *dominates* competition over customers, and the price is higher and the customers' average welfare is lower in the duopoly equilibrium than in the monopoly equilibrium.

6.1 Setup

The setup is similar to the monopoly setup, but with two firms. $\mathcal{F} = \{1, 2\}$ is the set of firms. When a firm f is clearly known from the context, we sometimes use the notation $-f$ to refer to the other firm.

The high-level description of the game is as follows. Each firm chooses price and wage. The *payment profile of firm* f is the tuple $\mathbf{P}_f = (p_f, w_f)$. The *payment profile* $\mathbf{P}$ is defined by the tuple $(\mathbf{P}_1, \mathbf{P}_2)$. The profit of each firm would then be determined by a subgame, in which agents (workers and customers) observe the payment profile and the decisions of the other agents and make (optimal) decisions based on that information.[10] In a *steady-state subgame equilibrium under payment profile* $\mathbf{P}$, no agent benefits from changing her decision, taking the decisions of other agents as given. We will see that any payment profile $\mathbf{P}$ *induces* an *essentially unique* steady-state subgame equilibrium. Then, a duopoly equilibrium will be defined as the equilibrium of a game played between the two firms whose actions are choosing price and wage.

The first step to formalize these definitions is defining the actions available to each agent.

Agents

Workers A worker can choose a subset of the firms to accept offers (ride requests) from. This gives her $2 \times 2 = 4$ possible actions, one of which she chooses. We say a worker takes action S if she chooses to accept offers from the subset $S \subseteq \mathcal{F}$ of the firms. In this case, we say the worker is *of type* S. We say a worker has *joined* firm f if she is of type S and $f \in S$.

Customers The model is similar to the monopoly model, with one main difference: customers' valuations are now modeled by a joint distribution over the two firms. We thereby suppose that a customer's valuations are represented by $(v_1, v_2) \sim G$, where v_f is the customer's valuation for firm f and $G : [0, 1]^2 \to [0, 1]$ denotes the CDF of the joint distribution.

[10] The decisions of all agents do not need to be common knowledge, so long as the values of some equilibrium parameters (such as size of the pool and the rate of customers who join) are.

Immediately upon her arrival, each customer takes one of the following actions: requesting service from firm 1 (i.e. *joining* firm 1), joining firm 2, or joining no firm. If the customer joins a firm, she will be served for a unit of time, after which she departs the market. If the customer decides to join no firm, she immediately departs the market.

Compositions

Worker composition A *worker composition* is a tuple (I, B) with

$$I = \{i(S) : \forall S \subseteq \mathcal{F}\},$$
$$B = \{b(S, f) : \forall f \in \mathcal{F}, S \subseteq \mathcal{F} \text{ such that } f \in S\},$$

where $i(S) \in \mathbb{R}_+$ denotes the mass of idle workers of type S and $b(S, f) \in \mathbb{R}_+$ denotes the mass of workers of type S busy at firm f. (It will become clear that these parameters have a steady-state interpretation.) We use $b(f)$ to denote the mass of all workers busy at firm f, i.e. $\sum_{S \ni f} b(S, f)$. We use $i(f)$ to denote the mass of all idle workers who accept offers from firm f, i.e. $\sum_{S \ni f} i(S)$.

Customer composition A *customer composition* is a tuple $\mathbf{k} = (k_1, k_2)$ where k_f denotes the (steady-state) rate of customers who join firm f.

Composition A composition $\mathbf{A}$ is a tuple $(\mathbf{k}, (I, B))$ where $\mathbf{k}$ is a customer composition and (I, B) is a worker composition.

Arrangement An arrangement is a tuple $\Sigma = (\mathbf{P}, \mathbf{A})$ where $\mathbf{P}$ is a payment profile and $\mathbf{A}$ is a composition.

Payoffs

Customer's payoff Under the arrangement $(\mathbf{P}, \mathbf{A})$, payoff of a customer from joining firm f is $v_f - p_f - c(i(f))$, where v_f is the valuation of the customer for firm f.

Worker's payoff Similar to the monopoly model, each worker has an outside option r, which is distributed from a distribution with CDF F. Given an arrangement $(\mathbf{P}, \mathbf{A})$, for any firm f, define $\gamma_f = \frac{k_f}{i(f)}$. The interpretation for γ_f is that it is the steady-state rate by which an idle worker who accepts offers from firm f receives offers from f. For any action

S, define $\gamma(S) = \sum_{f \in S} \gamma_f$. The interpretation for $\gamma(S)$ is that it is the steady-state rate by which a worker of type S receives offers. The payoff of a worker with outside option r who takes action S under arrangement $(\mathbf{P}, \mathbf{A})$ is

$$r \cdot \left(\frac{1}{1+\gamma(S)} \right) + \sum_{f \in A} w_f \cdot \frac{\gamma_f}{1+\gamma(S)}.$$

The interpretation of the above expression is that this is the steady-state earnings of a worker of type S per unit of time from wage and outside option. This expression is derived from a straight-forward exercise that computes the average time that a worker of type S remains idle or works at each of the firms (Lemma E.7).[11]

(Steady-state) subgame equilibrium

An arrangement $\Sigma = (\mathbf{P}, \mathbf{A})$ is said to be a *subgame equilibrium*,when the following conditions hold:

(i) **Customers optimize.** Each customer chooses the action that maximizes her payoff. In case the maximum payoff is attained by multiple actions, the customer chooses one of those actions uniformly at random.

(ii) **Workers optimize.** Each worker chooses the action that maximizes her payoff. In case the maximum payoff is attained by multiple actions, the worker chooses the action with the smallest size, i.e. if actions S, T both provide the maximum level of steady-state earnings, the worker prefers S to T if $|S| < |T|$.[12] If $|S| = |T|$ and S, T are the only actions that maximize the worker's steady-state earnings, then the worker chooses one arbitrarily.[13]

(iii) **Balance equations.** Actions taken by customers and workers *induce* $\mathbf{A}$ in the steady-state. Briefly, this means that the balance equations hold for all the parameters that define the composition $\mathbf{A}$. More precisely, assuming that the composition of workers and customers are determined by $\mathbf{A}$ at a given time, then (i) for any $S \in \mathcal{F}$, the in-flow

[11] When the workers of type S have a positive mass, i.e. when $b(S) + i(S) > 0$, the payoff of a worker of type S with outside option r could be written in a more intuitive form: $(1 - \sum_{f \in S} t_f) \cdot r + \sum_{f \in S} t_f \cdot w_f$, where t_f is the steady-state fraction of time that the worker is employed at firm f.

[12] It is possible to consider other rules, e.g. breaking the ties in favor of the action with the largest size; as we will discuss later, this will not change our results.

[13] By Fact E.8 in the appendix, this rule is never used, as its condition is never satisfied.

of idle workers of type S is equal to their out-flow, (ii) for any $S \in \mathcal{F}$ and $f \in \mathcal{F}$, the in-flow of workers of type S busy at firm f is equal to their out-flow, and (iii) the in-flow of customers at each firm is equal to their out-flow from that firm.

When $\Sigma = (\mathbf{P}, \mathbf{A})$ is a subgame equilibrium, we sometimes say that Σ is a *subgame equilibrium under* $\mathbf{P}$, or a *subgame equilibrium induced by* $\mathbf{P}$.

Definition 6.1. *A non-trivial subgame equilibrium is a subgame equilibrium in which both firms serve a positive rate of customers. A subgame equilibrium that is not non-trivial is called trivial.*

Our main focus is on the non-trivial equilibria, which, when exist, are uniquely determined by the payment profile.

Proposition 6.2. *Any payment profile* $\mathbf{P}$ *induces at most one non-trivial subgame equilibrium.*

Any payment profile $\mathbf{P}$ induces at least one trivial subgame equilibrium: the subgame equilibrium in which no firm serves any customers. We call this the $\varnothing$ subgame equilibrium. There are at most two other trivial subgame equilibria: for each firm, there is at most one subgame equilibrium at which only that firms serves a positive rate of customers. (Proposition E.9). As we will see soon, we focus on the non-trivial subgame equilibria for defining the notion of (global) duopoly equilibrium.

Definition 6.3. *The steady-state profit of a firm* f *in a subgame equilibrium* Σ *is*

$$\Pi_f(\Sigma) \equiv k_f \cdot (p_f - w_f),$$

where k_f, p_f, w_f *respectively denote the steady-state rate of customers who join* f*, and the price and wage at firm* f *in* Σ*.*

We are almost ready to define the duopoly equilibrium, that is, the equilibrium of the game played between the two firms. This will be our main equilibrium notion. In simple words, a duopoly equilibrium is a payment profile $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ such that no firm f can increase its profit by deviating from $\mathbf{P}_f$ to another payment profile $\overline{\mathbf{P}_f}$. There is, however, one subtlety. Firm f may choose $\overline{\mathbf{P}_f}$ so that the payment profile $\overline{\mathbf{P}} = (\overline{\mathbf{P}_f}, \mathbf{P}_{-f})$ induces no non-trivial equilibrium. In that case, which of the trivial equilibria induced by $\overline{\mathbf{P}}$ should be selected? There are several ways to address the multiplicity of trivial equilibria: (i) one can define a *selection rule* that selects one of the trivial equilibria in such cases, (ii) we can slightly modify the workers' tie-breaking rule (the rule that breaks the ties between their possibly multiple optimal actions) such that any payment profile would induce a unique subgame equilibrium,[14] and (iii) one can change the game that firms play by defining the firms' actions as quantity choices, rather than payment profiles (a well-known approach in the two-sided markets literature). All the above approaches lead to the same main insight, the adverse effect of competition in thin markets. (See the appendix, Subsection E.4.) To keep the main analysis simple, we take the first approach in here: defining a *selection rule*.

The selection rule

Given a payment profile $\mathbf{P}$, the selection rule selects a unique subgame equilibrium $\Sigma_{[\mathbf{P}]}$ that is induced by $\mathbf{P}$. We define $\Sigma_{[\mathbf{P}]}$ to be the steady-state non-trivial subgame equilibrium that serves the highest number of customers. We prove that $\Sigma_{[\mathbf{P}]}$ is in fact the (unique) non-trivial subgame equilibrium under $\mathbf{P}$, if one exists. Otherwise, $\Sigma_{[\mathbf{P}]}$ would be one of the trivial subgame equilibria under $\mathbf{P}$: the trivial subgame equilibrium under which the highest number of customers are served.[15] (Lemma E.11.)

Remark 6.4. *One of the simplest selection rules is the $\varnothing$ selection rule: the rule that selects the $\varnothing$ subgame equilibrium when a non-trivial subgame equilibrium does not exist. As the profits of both firms are 0 at the $\varnothing$ subgame equilibrium, this selection rule effectively eliminates deviations under which no non-trivial subgame equilibrium is induced by $\overline{\mathbf{P}}$. The selection rule that we chose does not eliminate such deviations, and therefore it gives a stronger equilibrium notion.*

Duopoly equilibrium

A subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is a *duopoly equilibrium* if for any firm f and any payment profile $\overline{\mathbf{P}} = (\overline{\mathbf{P}_f}, \mathbf{P}_{-f})$, $R_f(\Sigma_{[\mathbf{P}]}) \geq R_f(\Sigma_{[\overline{\mathbf{P}}]})$.

A duopoly equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is called *symmetric* iff $\mathbf{P}_1 = \mathbf{P}_2$.

Fact 6.5. *Both firms serve the same rate of customers at any symmetric duopoly equilibrium.*

[14] We present the main analysis under the tie-breaking rule that does not eliminate any of the trivial equilibria, with the intention of revealing their possible multiplicity.

[15] Notably, this coincides with the subgame equilibrium that maximizes customers' welfare, where welfare is defined in the usual way as the integral over customers' payoffs.

6.2 The adverse effect of competition

We start with an example. Let $c(x) = e^{-\gamma x}$ with $\gamma > 0$ and let F be the uniform distribution over the unit interval. Also, let G be the joint distribution for a customer's valuations over the two firms, i.e. $(v_1, v_2) \sim G$ is a customer's valuation for firms $1, 2$. We define G (implicitly) as follows:

$$v_1 = \sigma x + (1-\sigma)y,$$
$$v_2 = \sigma x + (1-\sigma)(1-y),$$

where x, y are iid uniform random variables with support over the unit interval and $\sigma \in (0,1)$. The interpretation is that x is the *common value component* and y is the *idiosyncratic component*. σ determines the correlation over customers' preferences. The higher σ, the higher $\operatorname{corr}(v_1, v_2)$ would be.

We compare the price and the customers' average welfare at the unique symmetric duopoly equilibrium to the price and the customers' average welfare at the unique monopoly equilibrium. (Whenever we refer to a monopoly or duopoly equilibrium in a formal statement, the proof for existence and uniqueness of the equilibrium will be included in the proof of that statement.) For brevity, we refer to the symmetric duopoly equilibrium as duopoly equilibrium, from now on.

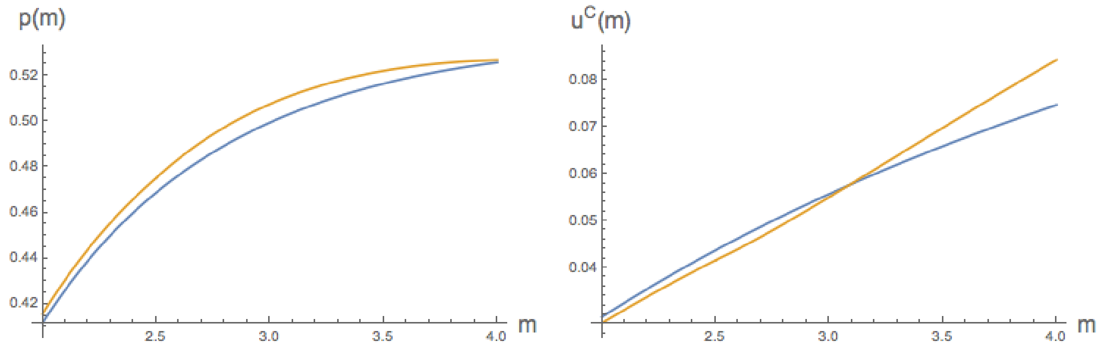
Let $p_{\text{duo}}(m)$ and $p_{\text{mon}}(m)$ respectively denote the equilibrium price at the duopoly and monopoly equilibria. Similarly, let $u^C_{\text{duo}}(m)$ and $u^C_{\text{mon}}(m)$ respectively denote the customers' average welfare at the duopoly and monopoly equilibria. Customers' average welfare is defined in the usual way, as the integral of payoffs over the customers who join divided by the rate of customers who join.[16]

In this example, there exist $\hat{m}_1$ such that $p_{\text{duo}}(m) > p_{\text{mon}}(m)$ holds iff $m < \hat{m}_1$. Similarly, there exists $\hat{m}_2$ such that $u^C_{\text{duo}}(m) < u^C_{\text{mon}}(m)$ holds iff $m < \hat{m}_2$. To demonstrate, we have plotted these quantities in Figure 6.

When the market is not sufficiently thick, the price is higher in the duopoly equilibrium and the customers' average welfare is lower. We call this the *adverse effect of competition*. Before generalizing the observation in this example, we explain the intuition. The same intuition holds in our more general statement. (Theorem 6.6)

The rough intuition is that there are two forces affecting the price. There is a *downward*

[16] In defining $u^C_{\text{duo}}(m)$, one may consider customers who join either of the firms, or only customers who join a fixed firm. The two definitions are identical, by symmetry of the duopoly equilibrium.



(a) $p_{\text{duo}}(m)$ and $p_{\text{mon}}(m)$

(b) $u^C_{\text{duo}}(m)$ and $u^C_{\text{mon}}(m)$

Figure 6: Price and customers' average welfare for $\sigma = 3/4$ and $\gamma = 1$.

force that pushes the price down, and is derived by competition over customers. There is also an *upward force* that pushes up the wage, and thereby price. The upward force is derived by competition over workers. The duopoly price is greater than the monopoly price when the upward force is stronger than the downward force, i.e. when competition over workers *dominates* competition over customers. This happens when the mass of potential workers, m, is not sufficiently large.

We will also provide a technical intuition by discussing the effects of competition over workers and customers on the firm's first-order condition. This will also explain what the term "dominates" precisely means. First, we extend the above observation to more general cost functions. We will let σ to be any constant greater than $1/2$, which ensures that the weight of the common value component is larger than the idiosyncratic component.

Theorem 6.6. *Let $\sigma > 1/2$. Then, there exists $\underline{m}$ such that monopoly and duopoly equilibria do not exist when $m \leq \underline{m}$. Moreover, there exists $\hat{m} > \underline{m}$ such that for all $m \in (\underline{m}, \hat{m})$, unique monopoly and duopoly equilibria exist and $p_{\text{duo}}(m) > p_{\text{mon}}(m)$ and $u^C_{\text{duo}}(m) < u^C_{\text{mon}}(m)$.*

Theorem 6.7. *Let $\sigma > 1/2$. Then, $w_{\text{duo}}(m) > w_{\text{mon}}(m)$ and $u^W_{\text{duo}}(m) > u^W_{\text{mon}}(m)$ hold at all m where monopoly and duopoly equilibria exist.*

In what follows, we provide a technical discussion on how the strength of competition over customers and workers is captured by a semi-elasticity term and a commission fee term that appear in the firm's first-order condition (FOC) for price.

We start by writing the firm's FOC in a familiar form. In the rest of the argument, we fix the price and wage offered by firm 2, and suppose that the function $\mathfrak{D} : [0,1]^2 \to [0,1]$ represents customers' demand for firm 1, i.e. $\mathfrak{D}(p, w)$ denotes the rate of customers who join the firm at price p and wage w. The firm's FOC for price could then be written as follows:

$$\left(-\frac{\mathfrak{D}_1(p,w)}{\mathfrak{D}(p,w)}\right)\cdot(p-w)=1. \tag{6.1}$$

Writing the firm's FOC as (6.1) has a certain appeal for our purpose. Note that the LHS is just the product of minus semi-elasticity of demand and the commission fee. The first factor reflects the strength of competition over customers and the second factor reflects the strength of competition over workers, as we will explain.

Let $p(k)$ and $w(k)$ denote the profit-maximizing price and wage for firm 1 conditioned on it serving a rate k of customers. We rewrite the LHS of (6.1) as a function of k, as follows. Define

$$\mathfrak{A}(k)\equiv\left(-\frac{\mathfrak{D}_1(p(k),w(k))}{k}\right)\cdot(p(k)-w(k)). \tag{6.2}$$

We would name $\mathfrak{A}(k)$ the *adjusted price elasticity of demand*, for reasons that are discussed next.

Let k^* denote the rate of customers that firm 1 serves in its profit-maximizing solution. We state two key facts without stating their proofs: (i) the condition $\mathfrak{A}(k^*)=1$ must hold, and (ii) $\mathfrak{A}(k)>1$ iff $k<k^*$. To understand these conditions better, note their similarity to the FOC of a hypothetical monopolist who produces a good with cost 0: it is well-known that the price elasticity of demand is -1 at the monopolist's optimal solution, which resembles (i) in our problem. Furthermore, under reasonable assumptions on the demand distribution, if the hypothetical monopolist sells to fewer customers (by posting a higher price), the elasticity would be smaller than -1. This resembles (ii) in our problem. Finally, observe that the RHS of (6.2) is similar to the expression for price elasticity of demand in the hypothetical monopolist's problem. The main difference is that second multiplicand in the definition of $\mathfrak{A}(k)$ is $p-w$ (rather than p), because in our problem there is a cost w per ride.

We are now ready to elaborate on the upward and downward forces that were discussed previously and precisely define what the term *dominates* means when we say the duopoly price is higher than the monopoly price when competition over workers dominates competition over customers. To this end, fix m, and consider a duopoly equilibrium in this market, which we call instance II. Construct instance I from instance II by removing firm 2 from the market and enforcing firm 1 to serve the same level of customers as in instance II. We are interested in understanding how adjusted price elasticity for firm 1 changes when we move from instance II to instance I. Let $\mathfrak{A}_{II}$ and $\mathfrak{A}_I$ respectively denote adjusted price elasticities in instances II and I, respectively. (Note that the expressions for $\mathfrak{A}_{II}, \mathfrak{A}_I$ involve different demand functions)

The commission fee term in $\mathfrak{A}_I$ is always larger than the commission fee term in $\mathfrak{A}_{II}$. We simply say that the commission fee goes up when firm 2 is removed. This is quite intuitive: the commission fee goes up when the competitor leaves. This is in fact the upward force that we discussed earlier. The commission fee goes up more when m is smaller, and so the upward force is stronger when m is smaller.

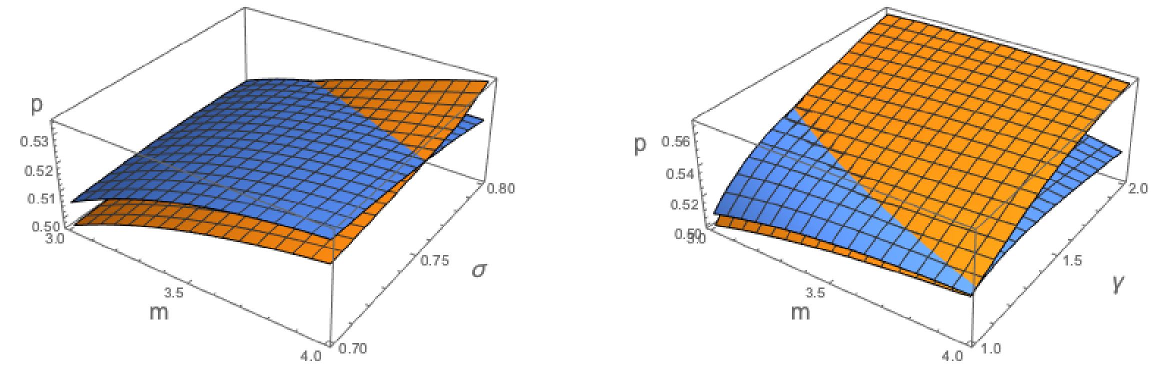
On the other hand, the semi-elasticity term in $\mathfrak{A}_I$ is smaller than the semi-elasticity term in $\mathfrak{A}_{II}$. We simply say that the semi-elasticity term goes down when firm 2 is removed. This is just the downward force that we discussed earlier. The semi-elasticity term goes down more when m is larger. The intuition is that at larger m, more customers with similar valuations for the two firms are served, and therefore, it becomes easier for a firm to attract customers (from the other firm) by lowering the price.[17] This creates a larger gap between the semi-elasticity terms when m is larger.

To sum up the intuition, observe that $\mathfrak{A}_{II} = 1$ always holds, as instance II is a duopoly equilibrium. When m is small, the upward force is strong and the downward force is weak, and thereby $\mathfrak{A}_I > \mathfrak{A}_{II} = 1$. Consequently, in instance I, firm 1 could increase profit by decreasing price and serving a larger rate of customers; this could result in firm 1 serving more customers in the monopoly equilibrium than in the duopoly equilibrium, and at a lower price.

6.3 Examples: correlation and matching technology

In Figure 7(a), we plot the monopoly and duopoly equilibrium prices while varying m and σ. (Recall that σ determines the correlation between a customer's preferences. The higher the σ, the higher the correlation.) For any fixed σ, we can observe the value of m at which the duopoly price is equal to the monopoly price. Denote this value by $m(\sigma)$, defining it is a function of σ. Therefore, for a fixed σ, the duopoly price is higher than the monopoly price iff $m < m(\sigma)$. The noteworthy point is that $m(\sigma)$ is decreasing in σ. This is observable in Figure 7(a), as $m(\sigma)$ is determined by the intersection of the two planes plotted in the figure. There is a simple interpretation: as σ goes up, competition over customers becomes stronger, and dominates the competition over workers at a lower value of m.

[17] Roughly speaking, a customer with valuation (v_1, v_2) values the two firms similarly if v_1, v_2 are "close".



(a) $m \in [3, 4]$ and $\sigma \in [0.7, 0.8]$ (b) $m \in [3, 4]$ and $\gamma \in [1, 2]$

Figure 7: The orange surface (lighter color) and blue surface (darker color) respectively give the monopoly and duopoly prices.

In Figure 7(b), the same exercise is done by replacing σ with γ. Similarly, for any fixed γ, let $m(\gamma)$ denote the value of m at which the duopoly price is equal to the monopoly price. Observe that that $m(\gamma)$ is decreasing in σ. The interpretation is that as γ goes up, competition over workers becomes weaker (because the matching technology of firms becomes stronger). Consequently, competition over customers dominates competition over workers sooner, i.e., at a lower value of m.

7 Discussion

In this section we discuss some of the assumptions that we make in our model, their role in the analysis, and also some the alternative modeling choices.

Alternative assumption: workers lose their outside option if they join the firm

In our main model a worker earns her outside option r whenever she is not busy serving a customer. An alternative assumption is that workers who join the firm (i.e. decide to accept ride requests from the firm) completely lose their outside option. Under this alternative assumption, a worker would join the firm iff her outside option is smaller than her steady-state income (from wage) per unit of time if she joins the firm.

The model under this alternative assumption is far less tractable. We qualitatively demonstrate that the same thin-market effects are still in play, and in fact, we observe that they are amplified. Figure 10 plots the equilibrium wage as a function of m under both assumptions. The left plot corresponds to the alliterative assumption (under which workers lose their outside option after joining) and the right plot corresponds to our original assumption. The non-monotonicity in wage is observed under both assumptions. Note that the scale of the horizontal axis in the left graph is almost twice its scale in the right, which suggests that the thin market complementarity effects persists more under the alternative assumption. The intuition is simple: under the alternative assumption, fewer workers join the firm at any fixed level of wage (because workers lose their outside option if they join); roughly speaking, this assumption makes the market effectively "thinner" and amplifies the thin-market effects.

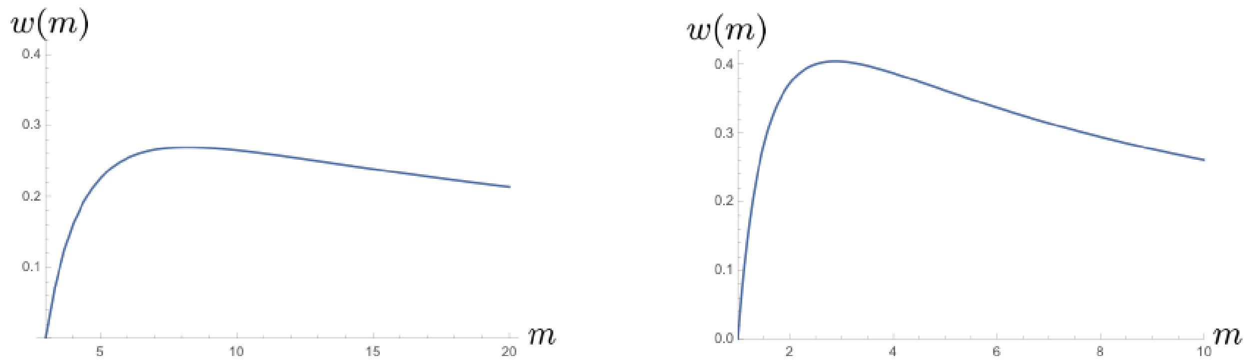


Figure 8: Equilibrium wages as functions of m for when F, G are the uniform distribution over $[0, 1]$ and $c(i) = e^{-i}$. The left and right plots respectively correspond to the alternative and original assumptions.

Convexity of the cost function c

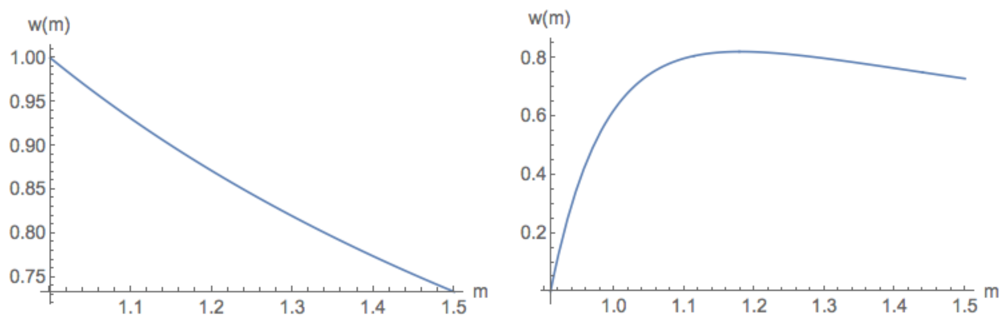
Convexity of the cost function is a main derivative of the thin-market complementarity effects. To clarify, we start with a monopoly example. Suppose that the cost function c belongs to the family

$$\mathcal{C} = \{c_\gamma : \gamma > 0\},$$

with $c_\gamma(i) \equiv (\max\{0, 1-i\})^\gamma$. Such functions are concave for $\gamma < 1$, affine for $\gamma = 1$, and convex for $\gamma > 1$. The main goal in this example is demonstrating the role of convexity of the cost function by letting γ vary from below 1 to above 1.

In Figure 9, we compare the the equilibrium wages as a function of m for when c is concave and convex. For $\gamma < 1$, the prediction by the law of demand holds: equilibrium wage decreases with the size of labor pool, m. For $\gamma > 1$, the equilibrium wage *increases* with m when m is below a certain threshold.

For the case of convex cost functions, we provided the intuition for non-monotonicity in the equilibrium wage in Section 4; recall that the intuition is based on the marginal revenue and marginal cost (per ride) of an increase in wage and the equality MR = MC which always holds at the monopoly equilibrium. This equality does not hold for concave cost functions because the monopoly solution would not be an interior solution of the firm's profit maximization problem, as demonstrated in A: for the case of uniform distributions F, G and concave cost functions c, the monopolist employs just enough workers to provide the lowest possible waiting time for customers, which would imply that the equilibrium wage decreases with the labor pool.



(a) Concave or affine c: any $\gamma \leq 1$. (b) Convex c: $\gamma > 1$

Figure 9

Next, we discuss that convexity of waiting times is a natural assumption in ride-sharing markets. Consider the following thought experiment: place i points in the unit circle independently uniformly at random. In this thought experiment, these points correspond to i idle workers available to serve a customer who is located on the center of the circle. The expected distance of the closest worker to the center of the circle is a convex, decreasing function of i. Similar measures have been defined and estimated empirically: the Expected Time of Arrival (ETA) of a driver as a function of the number of idle workers "around" a customer is estimated by ride-sharing platforms, such as Uber (e.g., see [Phillips 2017]). Although these estimates could vary across cities, convexity in the number of idle workers is their common feature.

The distribution of workers' outside options (F) has a decreasing PDF

Before discussing the role of this assumption, we remark that it is not a necessary assumption. Figure 10 demonstrates the non-monotonicity of wage in labor pool for when this assumption does not hold. To explain the role of this assumption, we first note that workers' average welfare may decrease when equilibrium wage increases; this could happen if, after an increase in wage, a "large" number additional workers join the firm, decreasing the average employment time and the average welfare of workers who work for the firm. The condition that F has a decreasing PDF essentially controls the number of additional workers who join the firm as the equilibrium wage goes up with the labor pool, and is sufficient for all of the theorems to hold.

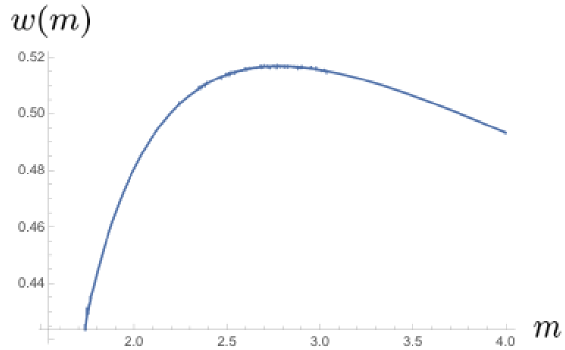


Figure 10: Equilibrium wage as a function of m for when the CDF of F is $f(x) = x^{3/2}$ for $x \in [0,1]$, G is the uniform distribution over $[0,1]$, and $c(i) = e^{-i}$.

References

M. Armstrong. Competition in two-sided markets. Industrial organization, EconWPA, 2005. URL http://EconPapers.repec.org/RePEc:wpa:wuwpio:0505009. 2, 3

M. Armstrong and J. Wright. Two-sided markets, competitive bottlenecks and exclusive contracts. *Economic Theory*, 32(2):353–380, Aug 2007. ISSN 1432-0479. doi: 10.1007/s00199-006-0114-6. URL https://doi.org/10.1007/s00199-006-0114-6. 4

S. Banerjee, C. Riquelme, and R. Johari. Pricing in ride-share platforms: A queueing-theoretic approach. 2015. 4

B. Caillaud and B. Jullien. Chicken & egg: Competition among intermediation service providers. *The RAND Journal of Economics*, 34(2):309–328, 2003. ISSN 07416261. URL http://www.jstor.org/stable/1593720. 3

J. C. Castillo, D. Knoepfle, and G. Weyl. Surge pricing solves the wild goose chase. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, EC '17, pages

241–242, New York, NY, USA, 2017. ACM. ISBN 978-1-4503-4527-9. doi: 10.1145/3033274.3085098. URL http://doi.acm.org/10.1145/3033274.3085098. 4

A. A. Cournot, I. Fisher, and N. T. Bacon. *Researches into the mathematical principles of the theory of wealth*. The Macmillan company, New York, 1927. URL //catalog.hathitrust.org/Record/001887291. 4

N. Economides. The economics of networks. *International Journal of Industrial Organization*, 1996. 4

N. Economides. Quality choice and vertical integration. *International Journal of Industrial Organization*, 17(6):903–914, 1999. URL https://EconPapers.repec.org/RePEc:eee:indorg:v:17:y:1999:i:6:p:903-914. 4

M. L. Katz and C. Shapiro. Network externalities, competition, and compatibility. *The American Economic Review*, 75(3):424–440, 1985. ISSN 00028282. URL http://www.jstor.org/stable/1814809. 4

W. M. W. Lam. Switching costs in two-sided markets. *The Journal of Industrial Economics*, 65(1):136–182, 2017. ISSN 1467-6451. doi: 10.1111/joie.12133. URL http://dx.doi.org/10.1111/joie.12133. 2

R. Phillips. Balancing supply and demand in a two-sided marketplace. *Presentation in Stanford University.*, 2017. 29

J.-c. Rochet and J. Tirole. Platform competition in two-sided markets. *Journal of the European Economic Association*, pages 990–1029, 2003. 3

O. Shy. A short survey of network economics. *Review of Industrial Organization*, 38(2):119–149, 2011. ISSN 0889938X, 15737160. URL http://www.jstor.org/stable/23884968. 4

J. Tirole and J. Rochet. Two-sided markets: A progress report. Post-print, HAL, 2006. URL http://EconPapers.repec.org/RePEc:hal:journl:hal-00173715. 3

E. G. Weyl. A price theory of multi-sided platforms. *American Economic Review*, 100(4):1642–72, September 2010. doi: 10.1257/aer.100.4.1642. URL http://www.aeaweb.org/articles?id=10.1257/aer.100.4.1642. 3

A. White and E. Weyl. Imperfect platform competition: A general framework. Working Papers 10-17, NET Institute, 2010. URL http://EconPapers.repec.org/RePEc:net:wpaper:1017. 3

Wikipedia. L'hopital's rule. https://en.wikipedia.org/wiki/L%27H%C3%B4pital%27s_rule#Corollary, July 2017. 49

Appendices

A The role of convexity

In this section, we first focus on a stylized family of cost functions, parameterized by γ, and defined by $c(i) = (\max\{0, 1-i\})^\gamma$. We also assume that F, G are the uniform distribution. The goal is highlighting the role of the convexity of the cost function in non-monotonicity of the equilibrium wage. Then, in Subsection A.1 we show that similar insights hold for general cost functions.

Affine cost function. Suppose $\gamma = 1$. First, see that $m_0 = 1$, because $p + c(mp) < 1$ implies $m > 1$. Fix some $m > m_0$. We investigate whether there exists a beneficial deviation $p' = p(m) + \epsilon_p$ and $w = w(m) + \epsilon_w$ which results in the same level of customers (but increases profit). Suppose we increase wage by ϵ_w; this results in a decrease in waiting cost with magnitude $m\epsilon_w$. Therefore, we can set $\epsilon_p = m\epsilon_w$. This results in the same level of customers, and increases profit per service. So, at the monopoly equilibrium, such a deviation should not be possible. That is, we must have $i(m) = 1$. In other words, when $\gamma = 1$, the monopolist always maintains the same level of idle workers, $i = 1$. The monopoly equilibrium then must be the solution to

$$\begin{aligned} &\max_{p,w\geq 0} \Pi(p,w) \\ \text{s.t. } & k^* = 1 - p^*, \\ & mw^* - k^* = 1. \end{aligned}$$

Solving by setting FOC to 0 implies that $w(m) = \frac{1+3m}{2m+2m^2}$, which is decreasing in m. The interpretation is simple: Let m marginally increase and ignore the changes in $k(m)$. If the level of wage stays the same, the number of idle workers will be larger than 1. The monopolist therefore takes wage down as m goes up.

Concave cost function. Suppose $\gamma < 1$. By the equation $c'(i(m)) = -1/m$, $i(m)$ must be decreasing in m. This means the force that pushes wage down as m goes up is even stronger than the linear case, which maintained the same level of idle workers. So, the equilibrium wage is decreasing in m in this case as well.

A.1 More general cost functions

We repeat the above exercise for more general cost functions and show that when c is concave or affine, equilibrium wage decreases with m. This is done by showing that in such cases, $c(i^*) = 0$ holds at any monopoly equilibrium. Given this fact, we can repeat the above exercise identically, which would imply that wage decreases with m.

Lemma A.1. *There is at most one monopoly equilibrium (p^*, w^*, k^*) that satisfies $c(i^*) = 0$.*

Proof. At a monopoly equilibrium, $c(i^*) = 0$ implies that $i^* = 1$. Therefore, $i^* = mw^* - k^*$ implies that $w^* = \frac{1+k^*}{m}$. This equation, together with the equation $k^* = 1 - p^* - c(i^*)$ and the FOC form a system of equations that characterize the equilibrium. This system has a unique solution, as shown in file "c0-eqm". □

Concave c. Suppose that c is strictly concave.

Lemma A.2. *The (unique) monopoly equilibrium (p^*, w^*, k^*) satisfies $c(i^*) = 0$.*

Proof. We prove that $c(i^*) = 0$ must be satisfied at any equilibrium. The proof of uniqueness then follows from (A.1). Proof by contradiction. We consider the deviation that increases both price and wage by $\epsilon > 0$, i.e. $p^\# = p^* + \epsilon$ and $w^\# = w^* + \epsilon$, and prove profit is decreasing along this direction. Define the function

$$\Pi_\epsilon(p, w) \equiv k(p + \epsilon, w + \epsilon) \cdot (p - w).$$

Observe that

$$\frac{\mathrm{d}\, \Pi_\epsilon(p^*, w^*)}{\mathrm{d}\, \epsilon} = \frac{\mathrm{d}\, k(p + \epsilon, w + \epsilon)}{\mathrm{d}\, \epsilon} \cdot (p - w), \tag{A.1}$$

$$\frac{\mathrm{d}^2\, \Pi_\epsilon(p^*, w^*)}{\mathrm{d}\, \epsilon^2} = \frac{\mathrm{d}^2\, k(p + \epsilon, w + \epsilon)}{\mathrm{d}\, \epsilon^2} \cdot (p - w). \tag{A.2}$$

Next, in file "concave-c" we compute

$$\begin{aligned} \frac{\mathrm{d}\, k(p^* + \epsilon, w^* + \epsilon)}{\mathrm{d}\, \epsilon} &= 0 \\ \frac{\mathrm{d}^2\, k(p^* + \epsilon, w^* + \epsilon)}{\mathrm{d}\, \epsilon^2} &= -\frac{(m+1)c''(mw^* - k)}{(c'(mw^* - k) - 1)^2} > 0, \end{aligned} \tag{A.3}$$

where the first inequality holds because the mass of customers who join, k, must not increase by deviation (ϵ, ϵ) and the second inequality holds because $c''(i) < 0$ for all $i > 0$. Now, (A.2) and (A.3) together imply that the deviation (ϵ, ϵ) increases the profit, which is a contradiction. □

Affine c. Suppose c is affine.

Lemma A.3. *The (unique) monopoly equilibrium (p^*, w^*, k^*) satisfies $c(i^*) = 0$.*

Proof. We prove that $c(i^*) = 0$ must be satisfied at any equilibrium. The proof of uniqueness then follows from (A.1). Consider an arbitrary monopoly equilibrium, namely (p, w, k) such that $mw - k > 0$. Also, consider an $\epsilon > 0$ sufficiently small. We show that the deviation that increases both price and wage by $\epsilon > 0$, i.e. $p^{\#} = p^* + \epsilon$ and $w^{\#} = w^* + \epsilon$ does not change the profit. The proof is by contradiction. Suppose it does. If the deviation increases the profit, then we reach a contradiction because we supposed (p, w, k) is a monopoly equilibrium. If the deviation $(+\epsilon, +\epsilon)$ decreases the profit, then the deviation $(-\epsilon, -\epsilon)$ must increase the profit. (This is a straight-forward consequence of affinity of c.) Therefore, the deviation (ϵ, ϵ) must not change the profit. This implies that, without changing the profit, we can change p^*, w^* by moving along the direction $(-\epsilon, \epsilon)$ until the number of idle workers is equal to 0. But at this point, the mass of customers who join, and thereby the profit, should be qual to 0. Contradiction. □

B Proofs from Subsection 4.2

First, we define some notation and go over some of the basic properties. These will be used in the proofs for Proposition 4.1 and Proposition 4.2.

B.1 Basic properties of the monopoly equilibrium

Lemma B.1. *The profit function $\Pi(p, w) \equiv (p - w) \cdot k(p, w)$ is bounded and continuous in (p, w).*

Proof. The proof follows from the market-clearing condition. □

Lemma B.2. *The partials $\Pi_p(p, w)$ and $\Pi_w(p, w)$ exist when $k(p, w) > 0$.*

Proof. See file "partials-uniformFG". □

The market equilibrium under one profit maximizing firm is called a *monopoly equilibrium*. Any non-binding monopoly equilibrium must satisfy the following conditions:

$$k = 1 - p - c(i), \tag{B.1}$$

$$k = -k_p(p, w) \cdot (p - w) \tag{B.2}$$

$$k = k_w(p, w) \cdot (p - w). \tag{B.3}$$

Recall that i denotes the number of idle workers, i.e. $i = mw - k$. (B.1) is just the market clearing condition. (B.2) and (B.3) are just the FOCs for price and wage, where k_p, k_w denote the partial derivatives of $k(p, w)$ with respect to p, w, respectively. In other words, (B.2) and (B.3), are just equations $\Pi_p(p, w) = 0$ and $\Pi_w(p, w) = 0$, rearranged.

Implicit differentiation from (B.1) implies

$$k_p(p, w) = \frac{1}{c'(mw - k) - 1}, \tag{B.4}$$

$$k_w(p, w) = \frac{mc'(mw - k)}{c'(mw - k) - 1}. \tag{B.5}$$

(B.2) and (B.3) imply that $k_p = -k_w$, which together with the above two equations implies

$$c'(mw - k) = \frac{-1}{m}. \tag{B.6}$$

(B.6) is an important condition. Intuitively, it is saying that for a marginal increase in wage, the monopolist can increase price by the same amount without changing the rate of customers who join.[18] Another way of writing (B.6) is $-c_w(m, w, k) = 1$, which could be interpreted as the equation MR = MC, i.e. the marginal revenue per ride for an increase in wage is equal to its marginal cost. The intuition that we provide in Subsection 4.2 is based on this interpretation. Next, we use (B.6) to simplify the equilibrium conditions.

By (B.6), (B.4) and (B.6), we can write

$$-k_p(p, w) = k_w(p, w) = \frac{m}{m+1}.$$

This allows us to rewrite the system of equations given by (B.1), (B.2), (B.3) as

[18] In other words, the marginal rate of substitution between price and minus wage is equal to 1 on any iso-quant of the customer's payoff function, which is defined as $u(v) = v - p - c(i)$, for a customer with valuation v.

$$k = 1 - p + c(i), \tag{B.7}$$

$$k = \frac{m}{m+1} \cdot (p - w), \tag{B.8}$$

$$c'(i) = \frac{-1}{m}. \tag{B.9}$$

Proposition B.3. *Any non-binding solution must satisfy*

$$\begin{cases} k = 1 - p + c(i), \\ k = \frac{m}{m+1} \cdot (p - w), \\ c'(i) = \frac{-1}{m}. \end{cases} \tag{B.10}$$

.

Proposition B.4. *Let $m_0 = \frac{-1}{c'(0)}$. There exists a solution (p, w, k) to (B.10) with $k \geq 0$ iff $m \geq m_0$. Furthermore, the solution is unique. Let $(p(m), w(m), k(m))$ denote the unique solution as a function of m. The functions $p(m), w(m), k(m) : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and differentiable at all $m > m_0$.*

Proof. Because c is convex, its derivative is increasing. Therefore, $c'(i) \geq c'(0)$ must always hold, which implies that $\frac{-1}{m} \geq c'(0)$ should hold if (B.10) has a solution. That is, $m \geq -\frac{1}{c'(0)}$ should hold. For the rest of the proof, see file "exists-uniformFG", where we find a closed-form solution for (B.10) when $m \geq -\frac{1}{c'(0)}$ and observe its differentiability when $m > -\frac{1}{c'(0)}$. □

We use a similar notation to denote the equilibrium values of other parameters, e.g. $\lambda(m)$ would denote the equilibrium value of λ as a function of m. By the usual convention, $p'(m), w'(m), k'(m)$ denote the derivatives of the equilibrium values with respect to m. When the argument m is clearly known from the context, we sometimes denote the equilibrium values by an asterisk, e.g. $w(m)$ would be denoted by w^*.

B.2 Proofs from Subsection 4.2

Proof of Proposition 4.1. The claims about uniqueness and existence are proved in Proposition B.4. The equation $c'(i^*) = -1/m$ readily implies

$$w^* = \frac{k^* + c'^{-1}(\frac{-1}{m})}{m}.$$

We will take the derivative with respect to m from both sides of this equality. First observe that

$$\frac{\mathrm{d}\, c'^{-1}(\frac{-1}{m})}{\mathrm{d}\, m} = \frac{1}{c''(w^*m - k^*)} \cdot \frac{1}{m^2}, \tag{B.11}$$

$$k'(m) = \frac{p^* + w^*m}{2(1+m)^2}, \tag{B.12}$$

where (B.12) is proved in file "proof-1.m". Using the above equations, we can then write

$$w'(m) = \frac{m \cdot \left(\frac{1}{c''(w^*m-k^*)} \cdot \frac{1}{m^2} + k'(m)\right) - w^*m}{m^2} \Rightarrow$$

$$w'(m) > 0 \Leftrightarrow \frac{1}{m^2 c''(w^*m - k^*)} + k'(m) > w^*, \tag{B.13}$$

and therefore,

$$w'(m) > 0 \Leftarrow \frac{1}{m^2 c''(w^*m - k^*)} > w^*. \tag{B.14}$$

which holds because (B.12) implies that $k'(m) > 0$. Finally, observe that

$$c_{m,w}(m, w^*, k^*) = c'(w^*m - k^*) + mw^*c''(w^*m - k^*)$$

$$= \frac{-1}{m} + mw^*c''(w^*m - k^*) \Rightarrow \tag{B.15}$$

$$c_{m,w}(m, w^*, k^*) < 0 \Rightarrow w^* < \frac{1}{m^2 c''(w^*m - k^*)}. \tag{B.16}$$

□

(B.16) Together with (B.14) prove the claim.

Proof of Proposition 4.2. The proof directly follows from (B.13). Multiplying both sides by

$\frac{m}{w^*}$ and rearranging the terms, we can rewrite the right-hand side of (B.13) as follows:

$$
\begin{aligned}
&\frac{1}{m^2 c''(w^* m - k^*)} + k'(m) > w^* \\
\Leftrightarrow\ &\frac{1}{m w^* c''(w^* m - k^*)} > \frac{(w^* - k'(m)) \cdot m}{w^*} \\
\Leftrightarrow\ &m w^* c''(w^* m - k^*) < \frac{w^*}{(w^* - k'(m)) \cdot m} \\
\Leftrightarrow\ &m w^* c''(w^* m - k^*) - \frac{1}{m} < \frac{w^*}{(w^* - k'(m)) \cdot m} - \frac{1}{m} \\
\Leftrightarrow\ &c_{m,w}(m, w^*, k^*) < \frac{w^*}{(w^* - k'(m)) \cdot m} - \frac{1}{m} \qquad \text{(B.17)} \\
\Leftrightarrow\ &c_{m,w}(m, w^*, k^*) < (w^* - k'(m)) \cdot \frac{m}{w^*} - \frac{1}{m} = \frac{k'(m)}{m \cdot (w^* - k'(m))}
\end{aligned}
$$

where (B.17) follows from (B.15).

□

As we saw earlier, the condition $c_{m,w}(m, w, k) < 0$ is equivalent to $\lambda c''(i) + c'(i) < 0$, which could be written as $\frac{c''(i)}{-c'(i)} \cdot \lambda > 1$. This could be interpreted as an elasticity condition: this is the partial elasticity of the marginal revenue per ride with respect to the mass of viable workers, λ. To see this more clearly, observe that $-c'(i)$ is the marginal amount by which the monopolist could raise the price without gaining or losing any customers for a marginal increase in λ, and that $c''(i)$ is just equal to the partial $\frac{\partial c'(\lambda - k)}{\partial \lambda}$, which is with respect to λ and holds k constant. This argument is formalized below.

Corollary B.5 (of Proposition 4.1)**.** $w'(m) > 0$ *holds if the partial elasticity of* $c'(i)$ *with respect to the mass of viable workers evaluated at the monopoly equilibrium is greater than* -1*, i.e.*

$$\frac{\partial c'(\lambda - k)}{\partial \lambda}\bigg|_{(\lambda^*, k^*)} > -1.$$

Furthermore, the above condition is equivalent to $c_{m,w}(m, w^*, k^*) < 0$.

Proof.

$$\begin{aligned}
&c_{m,w}(m,w^*,k^*) = c'(w^*m-k^*) + mw^*c''(w^*m-k^*) = \frac{-1}{m} + mw^*c''(w^*m-k^*) \\
&\Rightarrow \left(c_{m,w}(m,w^*,k^*) < 0 \Leftrightarrow w^* < \frac{1}{m^2c''(w^*m-k^*)}\right) \\
&\Rightarrow \left(c_{m,w}(m,w^*,k^*) < 0 \Leftrightarrow 1 < \frac{1}{w^*m^2c''(w^*m-k^*)}\right) \\
&\Rightarrow \left(c_{m,w}(m,w^*,k^*) < 0 \Leftrightarrow 1 < \frac{-c'(w^*m-k^*)}{w^*mc''(w^*m-k^*)}\right) \\
&\Rightarrow \left(c_{m,w}(m,w^*,k^*) < 0 \Leftrightarrow -1 < \frac{\partial c'(\lambda-k)}{\partial\lambda}\Big|_{(\lambda^*,k^*)}.\right)
\end{aligned}$$

Applying Proposition 4.1 completes the proof. □

B.3 Relaxing the distributional assumptions for the sufficient condition

In here, we dismiss the uniformity assumption and show that Proposition 4.1 would still hold for a broad class of distributions F, G. Let v^* denote the valuation of the customer who is indifferent between joining and not joining the firm, i.e. $v^* = p^* + c(i^*)$.

Proposition B.6. *Suppose that a monopoly equilibrium exists at m. If*

$$c_{mw}(m,w^*,k^*) < 0,$$

$$F''(w^*) \leq 0, \tag{B.18}$$

$$G''(v^*) \geq 0, \tag{B.19}$$

then $w'(m)$ exists and $w'(m) > 0$.

Condition (B.18) just says that the PDF of F should be decreasing at w^*. Both conditions (B.18) and (B.19) could be replaced with more relaxed conditions that ensure F is not "too convex" at w^* and G is not "too concave" at v^*; their current form, however, makes the proof simpler. In particular, condition (B.19) could be replaced with $\frac{-G''(v^*)}{G'(v^*)^2} < \frac{2}{1-G(v^*)}$. (See Proposition B.12) This condition is satisfied by a wide family of distributions, including exponential distributions and heavy-tailed distributions with finite mean.[19] [20]

[19] Heavy-tailed distributions are distributions defined over $[1,\infty)$ with CDF F such that $F(x) = 1 - x^{-\alpha}$. They have a finite mean when $\alpha > 1$

[20] Proposition B.6 does not need to bound the supports of F, G. The relaxed condition on G, however, is

To prove Proposition B.6, we first need a few preliminary results. An argument similar to Subsection B.1 lets us write the equilibrium characterizing conditions. We start with the following two lemmas.

Lemma B.7. *The profit function* $\Pi(p, w) \equiv (p - w) \cdot k(p, w)$ *is continuous and bounded by* 1 *at any* (p, w) *where* $k(p, w) > 0$.

Proof. The proof is followed from the market-clearing condition. □

Lemma B.8. *The partials* $\Pi_p(p, w)$ *and* $\Pi_w(p, w)$ *exist when* $k(p, w) > 0$.

Proof. See file "partials-relaxedFG". □

Lemma B.7 and Lemma B.8 allow us to write the equilibrium-characterizing condition as follows. (The steps are similar to the derivation of the equilibrium characterizing conditions Equation B.10 in Subsection B.1)

Proposition B.9. *Any non-binding solution must satisfy*

$$\begin{cases} k = 1 - G(p + c(mF(w) - k)), \\ k = -k_p(p, w) \cdot (p - w), \\ c'(i) = \frac{-1}{mF'(w)}. \end{cases} \tag{B.20}$$

.

It is also helpful to write the equilibrium-characterizing equations in a different form: in terms of allocation quantities rather than prices. To this end, suppose that $H \equiv F^{-1}$ and $J \equiv G^{-1}$. Also, recall that $\lambda = mF(w)$ denotes the mass of viable workers, and k denotes the rate of customers who join the firm. The firm's problem would then be choosing the quantities λ, k so that its profit is maximized, while the condition $k = 1 - G(p + c(\lambda - k))$ is satisfied. Observe that this equation allows us to write p in terms of k, λ as follows: $p = J(1 - k) - c(\lambda - k)$. Also, observe that $w = H(\frac{\lambda}{m})$.

We therefore can write the firm's profit function as

$$\Pi(\lambda, k) = k \cdot (J(1 - k) - c(\lambda - k) - H(\lambda/m)).$$

This allows us to write the firm's FOC for the choice of k:

$$k \cdot (c'(\lambda - k) - J'(1 - k)) - c(\lambda - k) - H(\lambda/m) + J(1 - k) = 0, \tag{B.21}$$

also satisfied by, e.g, truncated exponential and truncated heavy tail distributions.

which is obtained by setting the partial derivative of the profit function with respect to k to 0, i.e. setting $\Pi_k(\lambda, k) = 0$.

Furthermore, we can rewrite the equation $c'(i) = \frac{-1}{mF'(w)}$ as

$$c'(\lambda - k) + \frac{H'(\lambda/m)}{m} = 0. \tag{B.22}$$

We have shown that any monopoly equilibrium must satisfy (B.21) and (B.22).

Lemma B.10. *Any monopoly equilibrium satisfies* (B.21) *and* (B.22)*.*

We are now ready to prove Proposition B.6.

Proof of Proposition B.6. First, we prove that the monopoly equilibrium exists in an open interval around m. The proof uses the Implicit Function Theorem. We use the fact that the monopoly equilibrium at m satisfies the system of equations given by (B.21) and (B.22), and then apply the Implicit Function Theorem on this system by considering one independent variable, m, and two dependent variables, k, λ, whose values depend on m. Applying the theorem implies the existence of unique functions $k : \mathbb{R}_+ \to \mathbb{R}_+$ and $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ that determine the equilibrium values of k, λ in an open interval around m. Furthermore, it implies that the partials of the functions k, λ with respect to the independent variable exist in the open interval. The proof is in file "existence-Fcav-Gvex", where we show that the Implicit Function Theorem applies by showing that the corresponding Jacobian is invertible.

Claim B.11. $w'(m) > 0 \Leftrightarrow \lambda'(m) > F(w^*)$

Proof.

$$\begin{aligned}
&\lambda'(m) = mw'(m)F'(w^*(m)) + F(w^*(m))) > F(w^*(m)) \\
\Leftrightarrow\ & mw'(m)F'(w^*(m)) > 0 \\
\Leftrightarrow\ & w'(m) > 0.
\end{aligned}$$

□

To prove the main claim, we also we rewrite the "sufficient condition", i.e. the condition $c_{mw} < 0$, as

$$c_{m,w}(m, w, k) = c'(i) + \lambda c''(i) < 0, \tag{B.23}$$

where $\lambda = mF(w)$ denotes the number of viable workers.

The rest of the proof is done in file "wage-Fcav-Gvex". In this file, we compute $\lambda'(m)$ by implicit differentiation with respect to m from (B.21) and (B.22). Then, we prove that when the condition (B.23) and the rest of the given conditions in the statement are satisfied, $\lambda'(m) > F(w^*)$ holds, and therefore we have $w'(m) > 0$ by Claim B.11. □

Proposition B.12. *Suppose that*

$$\begin{aligned} c_{mw}(m, w^*, k^*) &< 0, \\ F''(w^*) &\leq 0, \\ \frac{-G''(v^*)}{G'(v^*)^2} &< \frac{2}{1 - G(v^*)}. \end{aligned}$$

Then, $w'(m) > 0$

Proof. See the file "wage-Fcav-G". □

Corollary B.13 (of Proposition B.6 and Proposition B.12)**.** *Suppose that* $F''(w^*) \leq 0$ *and* G *has a concave PDF. Then,*

$$c_{mw}(m, w^*, k^*) < 0 \Rightarrow w'(m) > 0$$

Proof. Observe that because of Lemma B.14, Proposition B.12 is applicable if $G''(v^*) < 0$. If $G''(v^*) \leq 0$, then we can apply Proposition B.6. □

Lemma B.14. *Suppose the CDF* G *has a concave and decreasing PDF. Then, for any* $v \in supp(G)$*,*

$$\frac{-G''(v)}{G'(v)^2} < \frac{2}{1 - G(v)}.$$

Proof. Let g denote the PDF of G. Since g is concave and decreasing, the bound

$$1 - G(v) < \frac{1}{2} \cdot \frac{g(v)}{-g'(v)} \cdot g(v)$$

always holds. Given this bound, we prove the claim of the lemma:

$$\frac{-G''(v)}{G'(v)^2} < \frac{2}{1-G(v)} \Leftarrow \frac{-G''(v)}{G'(v)^2} \cdot (1-G(v)) < 2$$
$$\Leftarrow \frac{-g'(v)}{g(v)^2} \cdot \frac{1}{2} \cdot \frac{g(v)}{-g'(v)} \cdot g(v) < 2$$
$$\Leftarrow 1 < 4.$$

□

B.4 The effect of thickness on the welfare of customers

For uniform F, G, $k(m)$ always increases with m . This is shown in file "k-cFG-uniformFG". Next, we show that this observation is fairly general, in the sense that it holds under reasonable assumptions on F, G.

Lemma B.15. *Suppose $f : [0,1] \to \mathbb{R}_+$ is a concave increasing function. Let $F(a) \equiv \int_0^a f(x)\,\mathrm{d}\,x$. Then, the function*

$$g(a) = \frac{\int_0^a (a-x)\cdot f(x)}{F(a)}$$

is increasing in a.

Proof. First, observe that

$$g'(a) = 1 - \frac{f(a)\int_0^a (a-x)f(x)\,dx}{F(a)^2}.$$

To prove the main claim, we will show that $g'(a) > 0$, or equivalently,

$$F(a) > \frac{f(a)\int_0^a (a-x)f(x)\,dx}{F(a)}$$
$$= af(a) - \frac{f(a)}{F(a)} \cdot \int_0^a xf(x)\,\mathrm{d}\,x,$$

which we write as

$$\frac{f(a)}{F(a)} \cdot \int_0^a xf(x)\,\mathrm{d}\,x > af(a) - F(a). \tag{B.24}$$

Let us denote the LHS and the RHS of the above inequality by $L(a)$ and $R(a)$, respectively. To prove that (B.24) holds for all $a \in (0,1)$, we prove that (i) $L'(a) - R'(a) > 0$ for all $a \in (0,1)$, and (ii) $\lim_{a\to 0} L(a) - R(a) \geq 0$. This would prove the main claim.

To see why (i) holds, observe that

$$L'(a) - R'(a) = \frac{\left(a\int_0^a f(x)\,dx - \int_0^a xf(x)\,dx\right)\left(f(a)^2 - f'(a)\left(\int_0^a f(x)\,dx\right)\right)}{\left(\int_0^a f(x)\,dx\right)^2}.$$

Concavity of f implies that the RHS of the above equality is positive.

To prove (ii), we use L'Hôpital's Rule and compute

$$\lim_{a\to 0} L(a) - R(a) = \lim_{a\to 0} \frac{f'(a)\left(\int_0^a xf(x)\,dx\right)}{f(a)} + af(a)$$

Observing that the RHS is non-negative concludes the claim. □

B.4.1 Uniform F, relaxed G

We show that $k(m)$ is always increasing in m if (i) G has a concave and decreasing PDF. (ii) G has an increasing PDF. These are of course only two cases that we can provide a fairly short proof for, and we expect this to hold more generally.

The proof starts in file "k-cFG-uniformF", where we see that

$$k'(m) > 0 \Leftrightarrow J''(1-k) < \frac{2J'(1-k)m + 2}{km},$$

where $J \equiv G^{-1}$. We write the above condition in terms of G as follows

$$\begin{aligned} k'(m) > 0 &\Leftrightarrow \frac{-G''(v)}{G'(v)^3} < \frac{\frac{2m}{G'(v)} + 2}{km} \\ &\Leftrightarrow \frac{-G''(v)}{G'(v)^2} < \frac{2 + \frac{2G'(v)}{m}}{k}, \end{aligned} \tag{B.25}$$

where $v \equiv J(1-k)$.

First, observe that if $G''(v) > 0$, then (B.25) always holds, and therefore part (ii) of the claim is proved. In the rest of the argument, we show why part (i) holds. We show that

$k'(m) > 0$ holds if the following condition holds, which is stronger than (B.25):

$$\frac{-G''(v)}{G'(v)^2} < \frac{2}{k},$$

Note that $k = 1 - G(v)$, and therefore we can write the above condition as

$$\frac{-G''(v)}{G'(v)^2} < \frac{2}{1 - G(v)}. \tag{B.26}$$

Lemma B.14 states that a sufficient condition for the above inequality to hold is that g is concave and decreasing.

B.4.2 relaxed F, relaxed G

In this section, we give a sufficient condition for $k'(m) > 0$ for when F has a decreasing PDF and G satisfies the assumption of subsubsection B.4.1. According to "k-cFG-relaxedFG", a sufficient and necessary condition for $k'(m) > 0$ is

$$J''(1-k)$$
$$< \frac{2c''(i)H'(\frac{\lambda}{m})m^2 + c''(i)H''(\frac{\lambda}{m})km + 2c''(i)J'(1-k)m^3 + 2H'(\frac{\lambda}{m})H''(\frac{\lambda}{m}) + 2H''(\frac{\lambda}{m})J'(1-k)m}{c''(i)km^3 + H''(\frac{\lambda}{m})km}.$$

A lower bound for the RHS of the above inequality is

$$\frac{2c''(i)J'(1-k)m^3 + 2H''(\frac{\lambda}{m})J'(1-k)m}{c''(i)km^3 + H''(\frac{\lambda}{m})km},$$

which is obtained by removing some of the positive terms from the numerator in the RHS of the original inequality. There is a simple lower-bound for this term itself, using which we can rewrite the sufficient condition for $k'(m) > 0$ as

$$J''(1-k) < \frac{2c''(i)J'(1-k)m^3 + 2H''(\frac{\lambda}{m})J'(1-k)m}{c''(i)km^3 + H''(\frac{\lambda}{m})km} = \frac{2J'(1-k)}{k}.$$

Now, this condition is just the same condition as in (B.26). The rest of the previous analysis (and the same identical assumption for G) also applies.

C Proof of Theorem 4.3

This section contains the proof for Theorem 4.3. The proof contains some technical steps. To help readability, we have stated some of these steps as claims and lemmas, and have moved the proofs for some of them to the end of this section, in Subsection C.1.

Let $S(m)$ denote the system of equations given by (B.21) and (B.22), for a given m. Let the function $X(m, \lambda, k)$ and $Y(m, \lambda, k)$ denote the LHS of equations (B.21) and (B.22), respectively. Also, let $m_0 = \frac{1}{-c'(0)F'(0)}$; we will show that $\underline{m} = m_0$.

The proofs for the next lemmas and claims are given in Subsection C.1.

Lemma C.1. *There exists a monopoly equilibrium at m iff $m > m_0$.*

For the rest of the proof, it is helpful to extend the domains of the functions c, F, G so that: (i) their domains contain an interval $(-\epsilon, 0)$ for a $\epsilon > 0$, and (ii) c remains in $\mathbf{C^4}$ and strictly convex in the extended domain, (iii) F, G remain in $\mathbf{C^4}$ in the extended domain, and (iv) F' remains decreasing in the extended domain. It is straight-forward to verify that such an extension exists. We proceed the rest of the proof assuming that the domains of c, F, G are extended to satisfy properties (i)-(iv).

Claim C.2. *There exists an open interval $I = (m_1, m_2)$ containing m_0 such that $S(m)$ has a solution at any $m \in I$. Furthermore, there exist unique continuously differentiable functions $\lambda(m), k(m) : \mathbb{R}_+ \to \mathbb{R}$ such that $(\lambda(m), k(m))$ is a solution to $S(m)$, for any $m \in I$.*

Note that in the above claim, we are allowing the functions $\lambda(m), k(m)$ to have a possibly negative range.

Claim C.3. *The following relations hold:*

$$\lim_{m \to m_0} k'(m) = 0, \quad \lim_{m \to m_0} k''(m) > 0$$
$$\lim_{m \to m_0} \lambda'(m) > 0.$$

Furthermore, the limits

$$\lim_{m \to m_0} k'''(m), \lim_{m \to m_0} \lambda''(m), \lim_{m \to m_0} \lambda'''(m)$$

exist and are finite.

Corollary C.4. *There exists $m_3 > m_0$ such that $\lambda'(m) \neq 0$ for any $m \in (m_0, m_3)$.*

To prove Theorem 4.3, we will show that there exists $\hat{m} > m_0$ such that $w'(m) > 0$ and $e'(m) > 0$ hold for all $m \in (m_0, \hat{m})$. (Recall that $e(m) = \frac{k(m)}{\lambda(m)}$.) This will be done in Proposition C.5 and Proposition C.8. The inequalities $w'(m) > 0$ and $e'(m) > 0$ also imply that $(u^W)'(m) > 0$. The latter fact could be proved, e.g., by a straight-forward calculation of $(u^W)'(m)$. We omit this calculation here, and proceed with proving Proposition C.5 and Proposition C.8.

Proposition C.5. *Suppose $c(0) = 1$, $F''(r) \leq 0$ for all r in the unit interval. Then there exists a threshold $\hat{m}$ such that for all $m \in (m_0, \hat{m})$, $e'(m) > 0$.*

Proof. First, we prove the following claims.

Claim C.6. *There exists $m_4 > m_0$ such that $e'(m)$ exists at all $m \in (m_0, m_1)$.*

Proof. First, observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Because $\lambda(m) > 0$ for all $m > m_0$, and because $k(m), k'(m)$ and $\lambda'(m)$ exist and are finite for m sufficiently close to m_0 (by Claim C.3), $e'(m)$ is exists and is finite for m sufficiently close to m_0.

□

Claim C.7. $\lim_{m \to m_0} e(m) = 0$.

Proof. First, observe that

$$\lim_{m \to m_0} e(m) = \lim_{m \to m_0} \frac{k(m)}{\lambda(m)} = \lim_{m \to m_0} \frac{k'(m)}{\lambda'(m)}.$$

L'Hôpital's Rule is applicable here by Corollary C.4. Next, we compute

$$\lim_{m \to m_0} \frac{k'(m)}{\lambda'(m)} =$$

$$\lim_{m \to m_0} \Big(k\lambda c''(i) H''(F(w)) + m(k+\lambda)c''(i)H'(F(w)) - c'(\lambda)\left(\lambda H''(F(w)) + mH'(F(w))\right) - H'(F(w))^2 \Big)$$
$$\times \Big\{ \lambda H''(F(w)) \cdot \Big(k\left(c''(i) - J''(1-k)\right) + 2J'(1-k) \Big)$$
$$+ \; mH'(F(w)) \cdot \Big((k+\lambda)c''(i) - kJ''(1-k) + 2J'(1-k) \Big)$$
$$- \; 2c'(i) \cdot \Big(\lambda H''(F(w)) + mH'(F(w)) \Big) \Big\}^{-1},$$

where $H \equiv F^{-1}$, $J \equiv G^{-1}$ and, for notational brevity, we have denoted $i(m), k(m), w(m), \lambda(m)$ by i, k, w, λ, respectively. This equation is derived from implicit differentiation of the equilibrium-characterizing constraints with respect to m. We omit the algebraic details. The full derivation of this equation is in file "emp-relaxedFG-proof2.m". We can simplify the above equation using the fact that $c'(i) = -\frac{1}{mF'(w)}$. After this simplification, we use the fact that k, i, λ all approach 0 as m approaches m_0 to compute the limit

$$\lim_{m \to m_0} \frac{k'(m)}{\lambda'(m)} = 0.$$

The details of this calculation are given in file "emp-relaxedFG-proof2.m"; we omit these details in here. □

Continuity of $e(m)$ at m_0 is ensured by Claim C.7. The rest of the proof is as follows. We will show that $\lim_{m \to m_0} e'(m)$ exists and is positive. This would imply that $e'(m_0)$ must also exist, and in fact,

$$e'(m_0) = \lim_{m \to m_0} e'(m).$$

(This is a consequence of L'Hôpital's Rule. See, for example, [Wikipedia 2017] for a proof.) Once we have shown this, the proof is complete: because $e'(m_0) > 0$, then there must exists $\hat{m}$ such that $e'(m) > 0$ for $m \in [m_0, \hat{m}]$, i.e. $e(m)$ is an increasing function over $[m_0, \hat{m}]$.

First, observe that for all $m > m_0$ we have

$$e'(m) = \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

Therefore,

$$\lim_{m\to m_0} e'(m) = \lim_{m\to m_0} \frac{k'(m)\lambda(m) - \lambda'(m)k(m)}{\lambda(m)^2}.$$

The L'Hôpital's Rule is applicable because $\lim_{m\to m_0} k(m) = k(m_0) = 0$ and $\lim_{m\to m_0} \lambda(m) = \lambda(m_0) = 0$ hold, as we observed in Claim C.2 and its proof. Thereby, we can write

$$\begin{aligned}\lim_{m\to m_0} e'(m) &= \lim_{m\to m_0} \frac{k''(m)\lambda(m) - \lambda''(m)k(m)}{2\lambda(m)\lambda'(m)} \\ &= \lim_{m\to m_0} \frac{k'''(m)\lambda(m) + k''(m)\lambda'(m) - \lambda'''(m)k(m) - \lambda''(m)k'(m)}{2\lambda'(m)^2 + 2\lambda(m)\lambda''(m)}, \qquad \text{(C.1)}\end{aligned}$$

where (C.1) is due to a second application of L'Hôpital's Rule. We can simplify this equality further and write

$$\lim_{m\to m_0} e'(m) = \lim_{m\to m_0} \frac{k''(m)\lambda'(m)}{2\lambda'(m)^2} = \lim_{m\to m_0} \frac{k''(m)}{2\lambda'(m)} > 0, \qquad \text{(C.2)}$$

where the above relation holds by Claim C.3. This proves the promised claim. □

Proposition C.8. *Suppose $c(0) = 1$, $F''(r) \leq 0$ for all r in the unit interval. Then there exists a threshold $\hat{m}$ such that for all $m < \hat{m}$, $w'(m) > 0$.*

Proof. We show that $\lim_{m\to m_0^+} w'(m) > 0$ in "file wage-relaxedFG-proof.m". Continuity of $w'(m)$ then implies that there exists $\delta > 0$ such that $w'(m) > 0$ for $m \in [m_0, m_0 + \delta]$. This proves the claim.

□

C.1 Missing proofs for the technical claims and lemmas

Proof of Lemma C.1. We say that a pair (p, w) is *feasible at m* when $k(p, w) > 0$ holds when the size of labor pool is m. We say that m (as size of the labor pool) is *feasible* if there exists a pair (p, w) which is feasible at m. We will show that m is feasible iff $m > m_0$. This would prove the claim.

Claim C.9. *Suppose (p, p) is feasible for a fixed m. Then, there exists $w < p$ such that (p, w) is also feasible.*

Proof. Let $k = k(p, p)$. There exists $w < p$ such that $p + c(mF(w) - k) < 1$, by the continuity of c and F. Because G is continuous and atom-less, (p, w) is feasible. □

Next, we show that if $m = \frac{1}{-c'(0)F'(0)} + \delta$ for some $\delta > 0$, then m is feasible. We prove this by proving the existence of a feasible (p, w). To do this, we use Claim C.9 and show that there exists a p such that (p, p) is feasible; that is,

$$p + c(mF(p)) < 1. \tag{C.3}$$

We proceed by showing that there exists $p > 0$ satisfying (C.3).

Define the function $f(p) \equiv p + c(mF(p))$. Obseve that $f(0) = 1$. We next show that $f'(0) < 0$, which would guarantee the existence of a positive p (sufficiently close to 0) that satisfies (C.3):

$$\begin{aligned} &f'(p) = 1 + mF'(p)c'(mF(p)) \\ \Rightarrow &f'(0) = 1 + mF'(0)c'(0) = 1 + \left(\frac{1}{-c'(0)F'(0)} + \delta\right) F'(0)c'(0) = \delta F'(0)c'(0) < 0. \end{aligned}$$

The last inequality is implied by the assumption $F''(r) \leq 0$, which implies $F'(0) > 0$. Given that $f'(0) = 1$, $f'(0) < 0$ guarantees that there exists p arbitrary close to 0 such that $p + c(mF(p)) < 1$.

It remains to show that any $m \leq \frac{1}{-c'(0)F'(0)}$ is not a feasible mass of workers. The proof is by contradiction. Suppose m is a feasible of workers. Therefore, a monopoly equilibrium exists. Let (p^*, w^*, k^*) denote the equilibrium parameters, and let $i^* = mw^* - k^*$.The monopoly equilibrium must satisfy $c'(i^*) = \frac{-1}{mF'(w^*)}$. Then, observe that

$$c'(i^*) = \frac{-1}{mF'(w^*)} \leq \frac{-1}{\frac{1}{-c'(0)F'(0)} \cdot F'(0)} = c'(0).$$

But then, strict convexity of c would imply that $c'(i^*) = c'(0)$. Therefore, $i^* = 0$, which also implies that $k^* = 0$. Contradiction.

□

Proof of Claim C.2. The proof is based on the Implicit Function Theorem. As we mentioned earlier, we apply the theorem on the system given by (B.21) and (B.22). First, recall that (B.21) and (B.22) are given by

$$k \cdot (c'(\lambda - k) - J'(1 - k)) - c(\lambda - k) - H(\lambda/m) + J(1 - k) = 0,$$

and

$$c'(\lambda - k) + \frac{H'(\lambda/m)}{m} = 0.$$

Also, recall that the left-hand sides of (B.21) and (B.22) are respectively denoted by $X(m, \lambda, k)$, and $Y(m, \lambda, k)$. Next, we show that $(\lambda, k) = (0, 0)$ is a solution to $S(m_0)$, i.e. $X(m_0, 0, 0) = 0$, and $Y(m_0, 0, 0) = 0$.

$$0 \cdot (c'(0) - J'(1)) - c(0) - H(0) + J(1) = 0,$$

which holds because $c(0) = 1$, $H(0) = 0$, and $J(1) = 1$. Also,

$$c'(0) + \frac{H'(0)}{m_0} = 0,$$

which hols because $m_0 = \frac{-1}{F'(0)c'(0)}$ and $H'(0) = \frac{1}{F'(0)}$.

To apply the implicit function theorem, we need to prove that the Jacobian

$$J(m, k, \lambda) = \begin{pmatrix} \frac{\partial X(m,\lambda,k)}{\partial \lambda} & \frac{\partial X(m,\lambda,k)}{\partial k} \\ \frac{\partial Y(m,\lambda,k)}{\partial \lambda} & \frac{\partial Y(m,\lambda,k)}{\partial k} \end{pmatrix}$$

is invertible at point $(m, \lambda, k) = (m_0, 0, 0)$. This is done in file "existence-Fcav". The Implicit Function Theorem therefore applies, and there exist an open interval $I \ni m_0$ and unique continuously differentiable functions $\lambda(m), k(m)$ that solve the system $S(m)$ for all $m \in I$. Furthermore, the theorem implies the existence of $\lambda'(m)$ and $k'(m)$ for all $m \in I$.

□

Proof of Claim C.3. Because $\lambda(m), k(m)$ are continuously differentiable functions, then we have $\lim_{m \to m_0} k(m) = k(m_0)$ and $\lim_{m \to m_0} \lambda(m) = \lambda(m_0)$. On the other hand, $\lambda(m_0) = k(m_0) = 0$ holds, as we showed in the proof of Claim C.2. Using this fact, we prove the claim in file "relaxedFG-limits", where we use implicit differentiation with respect to m from the system $S(m)$ to compute the closed-form expressions for the derivatives and their limits.

□

Proof of Corollary C.4. This is a consequence of Claim C.3 and continuity of $\lambda'(m)$. □

D Proof of Theorem 5.1

The proof structure is similar to the proof of Theorem 4.3. First, we need some notation. We use $c_1 : \mathbb{R}_+^2 \to \mathbb{R}$ and $c_2 : \mathbb{R}+^2 \to \mathbb{R}$ to denote the partials of c with respect to its first and second argument, respectively.

Any monopoly equilibrium must satisfy the following equations

$$k \cdot (c_2(\gamma, \lambda - k) - J'(1-k)) - c(\gamma, \lambda - k) - H(\lambda/m) + J(1-k) = 0, \tag{D.1}$$

$$c_2(\gamma, \lambda - k) + \frac{H'(\lambda/m)}{m} = 0. \tag{D.2}$$

These equations are identical to (B.21) and (B.22), but adapted to the new notation for the cost function. We use the notation $S(m, \gamma)$ to refer to the above system of equations.

The next lemma shows that $\underline{m}_\gamma = \frac{1}{-c_2(\gamma,0)F'(0)}$.

Lemma D.1. *For any γ, there exists a monopoly equilibrium at m iff $m > \frac{1}{-c_2(\gamma,0)F'(0)}$.*

The proof is identical the proof of Lemma C.1. For the rest of the proof, we fix a level of matching technology γ_0 and prove the theorem statement for γ_0. For notational simplicity, let $m_0 = \underline{m}_{\gamma_0}$.

Next, we present two counterparts for Claim C.2 and Claim C.3 in the proof of Theorem 4.3.

Claim D.2. *There exist open intervals $I_m = (m_1, m_2)$ and $I_\gamma = (\gamma_1, \gamma_2)$ with $m_0 \in I_m$ and $\gamma_0 \in I_\gamma$ such that $S(m, \gamma)$ has a solution for any m, γ with $m \in I_m$ and $\gamma \in I_\gamma$. Furthermore, there exist unique continuously differentiable functions $\lambda(m, \gamma), k(m, \gamma) : \mathbb{R}_+ \to \mathbb{R}$ such that $(\lambda(m, \gamma), k(m, \gamma))$ is a solution to $S(m, \gamma)$, for any m, γ with $m \in I_m$ and $\gamma \in I_\gamma$.*

The proof for Claim D.2 is similar to the proof of Claim C.2 and is by applying the Implicit function theorem. We do not repeat the proof here. Note that in the above claim, we are allowing the functions $\lambda(m, \gamma), k(m, \gamma)$ to have a possibly negative range. We use the notation $\lambda_i(m, \gamma), k_i(m, \gamma)$ to denote the partials of the functions λ, k with respect to their i-th argument, for $i \in \{1, 2\}$. Furthermore, $k_{i,j}(m, \gamma)$ denotes the cross-partial derivative with respect to the i-th and j-th arguments (with possibly $i = j$). Also define $k_{i,j,k}(m, \gamma)$ similarly.

Claim D.3.

$$k_2(m_0, \gamma_0) = 0, \quad k_{2,2}(m_0, \gamma_0) > 0,$$
$$\lambda_2(m_0, \gamma_0) > 0.$$

Furthermore, the limits

$$\lim_{m \to m_0} k_{2,2,2}(m, \gamma_0), \lim_{m \to m_0} \lambda_{2,2}(m, \gamma_0), \lim_{m \to m_0} \lambda_{2,2,2}(m, \gamma_0)$$

exist and are finite.

Proof. See file "MT.m". □

To prove Theorem 5.1, we will show that there exists $\hat{m} > m_0$ such that $w_2(m, \gamma_0) > 0$ and $e_2(m, \gamma_0) > 0$ hold for all $m \in (m_0, \hat{m})$. This will be done in Proposition D.4 and Proposition D.5 (counterparts to Proposition C.5 and Proposition C.8).

The inequalities $w_2(m, \gamma_0) > 0$ and $e_2(m, \gamma_0) > 0$ also imply that $(u^W)_2(m, \gamma_0) > 0$. We proceed with proving Proposition D.4 and Proposition D.5.

Proposition D.4. *There exists a threshold $\hat{m}$ such that for all $m \in (m_0, \hat{m})$, $e_2(m, \gamma_0) > 0$.*

Proof. First, observe that for all $m > m_0$ we have

$$e_2(m, \gamma_0) = \frac{k_2(m, \gamma_0)\lambda(m, \gamma_0) - \lambda_2(m, \gamma_0)k(m, \gamma_0)}{\lambda(m, \gamma_0)^2}.$$

Therefore,

$$\lim_{m \to m_0} e_2(m, \gamma_0) = \lim_{m \to m_0} \frac{k_2(m, \gamma_0)\lambda(m, \gamma_0) - \lambda_2(m, \gamma_0)k(m, \gamma_0)}{\lambda(m, \gamma_0)^2}.$$

The L'Hôpital's Rule is applicable because the following hold: $\lim_{m \to m_0} k(m, \gamma_0) = k(m_0, \gamma_0) = 0$ and $\lim_{m \to m_0} \lambda(m, \gamma_0) = \lambda(m_0, \gamma_0) = 0$. (The proof is similar to the proof of Claim C.2).

Thereby, we can write

$$\lim_{m \to m_0} e_2(m, \gamma_0) \tag{D.3}$$

$$= \lim_{m \to m_0} \frac{k_{2,2}(m, \gamma_0)\lambda(m, \gamma_0) - \lambda_{2,2}(m, \gamma_0)k(m)}{2\lambda(m, \gamma_0)\lambda_2(m, \gamma_0)}$$

$$= \lim_{m \to m_0} \frac{k_{2,2,2}(m, \gamma_0)\lambda(m, \gamma_0) + k_{2,2}(m, \gamma_0)\lambda_2(m, \gamma_0) - \lambda_{2,2,2}(m, \gamma_0)k(m, \gamma_0) - \lambda_{2,2}(m, \gamma_0)k_2(m, \gamma_0)}{2\lambda_2(m, \gamma_0)^2 + 2\lambda(m, \gamma_0)\lambda_{2,2}(m, \gamma_0)}, \tag{D.4}$$

where (D.4) is due to a second application of L'Hôpital's Rule. We can simplify this equality further and write

$$\lim_{m \to m_0} e_2(m, \gamma_0) = \lim_{m \to m_0} \frac{k_{2,2}(m, \gamma_0)\lambda_2(m, \gamma_0)}{2\lambda_2(m, \gamma_0)^2} = \lim_{m \to m_0} \frac{k_{2,2}(m, \gamma_0)}{2\lambda_2(m, \gamma_0)} > 0, \tag{D.5}$$

where the above relation holds by Claim D.3. This proves the promised claim. □

Proposition D.5. *There exists a threshold $\hat{m}$ such that for all $m < \hat{m}$, $w_2(m, \gamma_0) > 0$.*

Proof. We show that $w_2(m_0, \gamma_0) > 0$ in file "MT.m". Continuity of the function $w_2(\cdot, \cdot)$ then implies that there exists $\delta > 0$ such that $w_2(m, \gamma_0) > 0$ for $m \in [m_0, m_0 + \delta]$. This proves the claim. □

E Preliminary results on duopoly

E.1 Notation

When referring to the customer composition of a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, we typically use the variable $\mathbf{k} = (k_1, k_2)$, when there is no risk of confusion. Furthermore, we usually use the variable k to denote $k_1 + k_2$. Similarly, we typically use the variables p_f, w_f to denote the price and wage at firm f in the payment profile $\mathbf{P}$.

Given Σ, we also define the notion of the *aggregate cost* that customers face at firm f to be $p_f + c(i_f)$, where i_f is the number of idle workers who accept offers from firm f in Σ. We typically use the variable b_f to denote $p_f + c(i_f)$.

In the analysis we sometimes consider a variable other than Σ, typically Σ', to denote a subgame equilibrium. In such cases, we will use a notation similar to the above notation; e.g., p'_f, w'_f will denote price and wage at firm f and k'_f will denote the rate of customers who join firm f in Σ'.

The firm's demand function The function $D : [0,1]^2 \to [0,1]$ is the *demand function* of customers. $D(b_1, b_2)$ determines the mass of customers demanding to join firm 1 assuming that the aggregate cost at firm f is b_f. By the symmetry of our model, $D(b_2, b_1)$ is the mass of customers demanding to join firm 2. This function has a simple geometric representation, depicted in Figure 11. Each point (x, y) in the unit square represents a customer with with valuation (v_1, v_2) defined as

$$v_1 = \sigma x + (1 - \sigma)y,$$
$$v_2 = \sigma x + (1 - \sigma)(1 - y).$$

Line l_1 corresponds to the customers who earn 0 payoff from joining firm 1. More precisely, l_1 is the line $\sigma x + (1 - \sigma)y = b_1$. Similarly, line l_2 corresponds to the customers who earn 0 payoff from joining firm l_2, i.e. the line $\sigma x + (1 - \sigma)(1 - y) = b_2$. The red shaded area is $D(b_1, b_2)$, and the blue shaded area is $D(b_2, b_1)$. The function D, obviously, has a closed-form expression in terms of b_1, b_2.

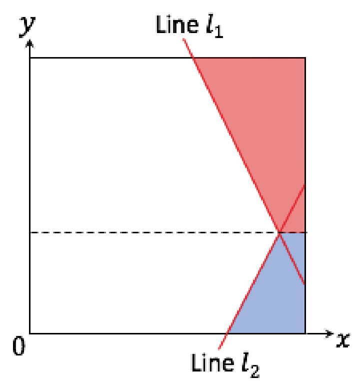


Figure 11: A Graphical representation for the demand function

E.2 Preliminaries

Lemma E.1. *In any non-trivial subgame equilibrium* $\Sigma = (\mathbf{P}, \mathbf{A})$*, there exists a cutoff* r^* *such that all workers with* $r < r^*$ *choose to accept offers from both firms and all workers with* $r \in (r^*, \max\{w_1, w_2\}]$ *choose to accept offers only from the firm with the maximum wage. If case* $w_1 = w_2$*, then* $r^* = w_1$*.*

Proof of Lemma E.1. If $w_1 = w_2$, the claim is proved because any individual worker who joins only one firm could strictly increase her payoff by joining both firms. Then, suppose without loss of generality that $w_1 > w_2$. Any worker who accepts offer from firm 2 in Σ should also accept offers from firm 1: if not, she can increase her payoff by accepting offers also from firm 1. Therefore, the set of workers could be partitioned into 3 subsets: workers of the *high* type, who accept offers only from firm 1, workers of the *low* type, who accept offers from both firms, and workers of the *null* type, who accept offers from no firm. It is straight-forward to see that a worker with outside option r is of the null type iff $r \geq w_1$.

First, we show that if a worker, namely worker 1, with option r_1 accepts offers from both firms, then any other worker, namely worker 2, with outside option $r_2 < r_1$ also accepts offers from both firms. Suppose that worker 1 switches to the strategy of accepting offers from firm 1 only. Let $\Delta_{[t_f]}$ denote the *additional* amount of time that worker will spend working at firm f after she switches. ($\Delta_{[t_f]} < 0$ means that the worker spends less time working at f) Similarly, let $\Delta_{[t_\varnothing]}$ denote the additional amount of time that the worker is unemployed after she switches. Because workers choose actions optimally, we must have

$$r_1 \cdot \Delta_{[t_\varnothing]} + w_1 \cdot \Delta_{[t_\varnothing]} + w_2 \cdot \Delta_{[t_\varnothing]} < 0.$$

On the other hand, it is straight-forward to observe that $\Delta_{[t_\varnothing]} > 0$, that is, worker 1 spends more time unemployed after her switch. Therefore, we should have

$$r_2 \cdot \Delta_{[t_\varnothing]} + w_1 \cdot \Delta_{[t_\varnothing]} + w_2 \cdot \Delta_{[t_\varnothing]} < 0.$$

This implies that worker 2 can increase her steady-state earnings if she accepts offers from both firms, which is a contradiction.

Now, let r^* be the infimum of r over all workers with outside option r who accept offers from firm 1. (Note that there exist such workers, because there is a positive mass of workers with outside option greater than w_2) This finishes the proof. □

Definition E.2. *The cutoff representation of a non-trivial subgame equilibrium* $\Sigma = (\mathbf{P}, \mathbf{A})$ *is given by* $(\mathbf{P}, r, \mathbf{k})$*, where*

(i) $\mathbf{k}$ *denotes the customer composition given by* $\mathbf{A}$

(ii) r *denotes the cutoff obtained by applying Lemma E.1 on* Σ*.*

Observe that if the cutoff representations of two non-trivial subgame equilibria are equal, then those subgame equilibria are equal.

Definition E.3. *The extended cutoff representation of a non-trivial subgame equilibrium* $\Sigma = (\mathbf{P}, \mathbf{A})$ *is given by* $(\mathbf{P}, r, \theta, \mathbf{k})$*, where*

(i) $(\mathbf{P}, r, \mathbf{k})$ *is the cutoff representation of* Σ*.*

(ii) $\theta = \frac{\lambda - mr}{k_{-f} + \lambda}$*, where* $f = \arg\max_{f \in \mathcal{F}} \{w_f\}$ *and* $\lambda = mw_f$*.*

Observe that $\theta = 0$ when $w_1 = w_2$. Variable θ has a simple interpretation: a fraction θ of the mass of idle workers accept offers only from the firm who offers the maximum wage. The value of θ (given in Definition E.3) is derived by solving the equation

$$mr = (\lambda - k)(1 - \theta) + k_f(1 - \theta) + k_{-f},$$

which computes the mass of workers who accept offers from both firms. The following fact uses the above equation to write r in terms of the rest of the parameters involved.

Fact E.4. *Let* $\Sigma = (\mathbf{P}, r, \theta, \mathbf{k})$ *be a non-trivial subgame equilibrium with* $w_1 \geq w_2$*. Then,*

$$r = \frac{(\lambda - k)(1 - \theta) + k_1(1 - \theta) + k_2}{m}.$$

Lemma E.5. *Let* $\Sigma = (\mathbf{P}, r, \theta, \mathbf{k})$ *be a non-trivial subgame equilibrium with* $w_1 \geq w_2$*. Then, we must have* $(1 - \theta)(mw_1 - k) = mw_2 - k$*.*

Proof. The proof is trivial when $w_1 = w_2$. Without loss of generality, suppose $w_1 > w_2$. Observe that r, θ are related by

$$r = \frac{(\lambda - k)(1 - \theta) + k_1(1 - \theta) + k_2}{m}, \tag{E.1}$$

which holds because exactly a fraction $1 - \theta$ of the busy workers at firm 1 and all the busy workers at firm 2 must be accepting offers from both firms.

Define $c_2 = c(i \cdot (1 - \theta))$. Now, observe that

$$\theta = 1 - \frac{c^{-1}(c_2)}{mw_1 - k}, \tag{E.2}$$

where recall that $i = \lambda - k$, by definition.

Next, we plug in (E.1) and (E.2) into worker's indifference condition and solve the equation for w_1. This lets us write w_1 as follows:

$$w_1 = \frac{c^{-1}(c_2) \cdot (k_2 - mw_1) + k_1 k_2 + (k_2 - mw_1)(k_2 - mw_2)}{k_1 m}$$

Rearranging the terms of the above equality implies

$$0 = (k_2 - mw_1)(c^{-1}(c_2) + k_1 + k_2 - mw_2).$$

Now, note that $k_2 < mw_2 \leq mw_1$ must always hold, and therefore, according to the above equality, we must have

$$c^{-1}(c_2) = mw_2 - k,$$

which implies $c_2 = c(mw_2 - k)$. ☐

Proposition E.6. *In any non-trivial subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, the waiting cost incurred by customers at firm f is $c(mw_f - k)$.*

Proof. The proof is a direct consequence of Lemma E.5. ☐

E.3 Missing proofs from Subsection 6.1

Lemma E.7. *Consider the following continuous-time stochastic process with state space $V = \{0, 1, 2\}$. The transition rate from state 0 to i is λ_i, for $i \in \{1, 2\}$. After a transition from state 0 to state i, the process remains at state i for a unit of time, after which it returns to state 0. Let π_i denote the fraction of time that the process spends at state i.*[21] *Then, $\pi_i = \frac{\lambda_i}{1+\lambda_1+\lambda_2}$ for $i > 0$ and $\pi_0 = \frac{1}{1+\lambda_1+\lambda_2}$.*

Proof. Straight-forward LLN arguments imply that $\pi_1 = \pi_2 \cdot \frac{\lambda_1}{\lambda_2}$ and $\pi_0(\lambda_1 + \lambda_2) = \pi_1 + \pi_2$. These two equations together with $\sum_{i=0}^{2} \pi_i = 1$ prove the claim. ☐

Fact E.8. *Given a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, there is no worker such that for that worker, there are only two actions $S = \{1\}, T = \{2\}$ that both provide her the maximum steady-state earnings.*

Proof. Proof by contradiction. Note that the worker's payoff under either action should be positive, because otherwise, the action $\varnothing$ attains the same level of steady-state earnings, which is a contradiction. Now, suppose that $w_2 \leq w_1$ without loss of generality. It is

[21] To state this more precisely, one should define $\pi_{i,t}$ to be the fraction of time that the process spends at state i from the start of the process until time t, and then define $\pi_i = \lim_{t\to\infty} \pi_{i,t}$.

straight-forward to see that the action $\{1, 2\}$ provides a larger level of steady-state earnings than the action $\{2\}$, which is a contradiction. $\square$

Proof of Proposition 6.2. The proof is by contradiction. Suppose there exist two non-trivial subgame equilibria, namely $\Sigma = (\mathbf{P}, \mathbf{A})$, and $\Sigma' = (\mathbf{P}, \mathbf{A}')$. Let $\mathbf{k}$ and $\mathbf{k}'$ respectively denote the customer compositions in $\mathbf{A}$ and $\mathbf{A}'$. Suppose $\mathbf{k} = (k_1, k_2)$ and $\mathbf{k}' = (k'_1, k'_2)$, and let $k = k_1 + k_2$ and $k' = k'_1 + k'_2$.

By Lemma E.5, the waiting cost that customers incur at firm f in Σ is equal to $c(mw_f - k)$. Let $b_f = p_f + c(mw_f - k)$ denote the aggregate cost that customers incur at firm f in Σ.[22] Define b'_f similarly for Σ'. Furthermore, with slight abuse of notation, we define the function $b_f : \mathbb{R} \to \mathbb{R}$ as

$$b_f(\hat{k}) = p_f + c(mw_f - \hat{k}),$$

where we have taken $\hat{k}$ to be a variable. Using this notation, we write the market clearing condition, which states that

$$\hat{k} = D(b_1(\hat{k}), b_2(\hat{k})) + D(b_2(\hat{k}), b_1(\hat{k})).$$

(Recall the definition of the demand function D from Subsection E.1.) In simple words, the condition says that the mass of busy drivers must be equal to the rate of customers who are served. Observe that the LHS is strictly increasing in $\hat{k}$, whereas the RHS is decreasing in $\hat{k}$. Therefore, this equation has a unique solution, which we denote by k^*. Note that we must have $k = k^*$ and $k' = k^*$. Therefore, $k = k'$. But then, this implies that $b_f = b'_f$ for all f (because $b'_f = p'_f + c(mw'_f - k')$, by definition). Therefore,

$$k_f = D(b_f(k), b_{-f}(k)) = k'_f$$

must hold for all f, which implies that $\mathbf{k} = \mathbf{k}'$. This, together with Lemma E.5 imply that the cutoff representations of Σ, Σ' are identical, which means that Σ, Σ' are identical. $\square$

Proposition E.9. *For any payment profile $\mathbf{P}$ and any firm $f \in \mathcal{F}$, there exists at most one trivial subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ satisfying $k_f > 0$.*

Proof. Suppose Σ is such a trivial subgame equilibrium, and let $\mathbf{k}$ denote the customer composition in $\mathbf{A}$. In Σ, all workers with outside option less than w must be accepting

[22] When we defined the notion of worker composition, we noted that $b(f)$ denotes the mass of workers busy at firm f. This notation is not used in this proof and is irrelevant to b_f.

the offers of firm f, and all other workers should be rejecting them. Therefore, any trivial subgame equilibrium in which f serves a non-zero rate of customers has the same worker composition as Σ.

On the other hand, see that the waiting cost that customers incur at firm f in Σ is equal to $c(mw_f - k)$. Recall the market-clearing condition (3.1), according to which we can write

$$k = 1 - p_f - c(mw_f - k)$$

for Σ. Observe that the LHS is strictly increasing in k, while the RHS is decreasing. Therefore, this equation has a unique root, k. This implies that any trivial subgame equilibrium in which f serves a non-zero rate of customers must have the same customer composition as Σ. This completes the proof. $\square$

E.3.1 The selection rule

First, we discuss an ascending process that finds the $\Sigma_{[\mathbf{P}]}$. We then present the missing proofs.

By Proposition E.6, the waiting cost that customers at firm f incur in $\Sigma = (\mathbf{P}, \mathbf{A})$ is equal to $c(mw_f - k)$. Let $b_f = p_f + c(mw_f - k)$. Further more, with slight abuse of notation, we define the function $b_f : \mathbb{R} \to \mathbb{R}$ as

$$b_f(\hat{k}) = p_f + c(mw_f - \hat{k}),$$

where we have taken $\hat{k}$ to be a variable. Using this notation, we write the market clearing condition

$$\hat{k} = D(b_1(\hat{k}), b_2(\hat{k})) + D(b_2(\hat{k}), b_1(\hat{k})). \tag{E.3}$$

Observe that the LHS is strictly increasing in $\hat{k}$, where as the RHS is decreasing in $\hat{k}$. Further more, the RHS is at most 1 at $\hat{k} = 0$, and is 0 at $\hat{k} = 1$. So, the above equation has a unique solution, which we denote by k^*.

Definition E.10. *Define the ascending process A^* as follows: the process starts at $\hat{k} = 0$ and increases $\hat{k}$ until* (E.3) *holds.*

Observe that the ascending process stops at $\hat{k} = k^*$. Given k^*, we define $\Sigma_{[\mathbf{P}]} = (\mathbf{P}, \mathbf{A})$ as follows:

1. The customer composition $\mathbf{k}$ satisfies $k_1 = D(b_1(k^*), b_2(k^*))$ and $k_2 = D(b_2(k^*), b_1(k^*))$. Furthermore, $k = k^*$, where recall that $k \equiv k_1 + k_2$ by definition.

2. If $k_f = 0$ for some firm f, then a non-trivial subgame equilibrium does not exist (see Lemma E.11). The selection rule chooses the trivial subgame equilibrium in which workers do not accept offers from firm f. Workers with outside option r accept offers of $-f$ iff $k_{-f} > 0$ and $r < w_{-f}$.

3. If $k_1, k_2 > 0$, then a non-trivial subgame equilibrium exists. Let $(\mathbf{P}, r^*, \theta, \mathbf{k})$ denote the extended cutoff representation of $\Sigma_{[\mathbf{P}]}$. Workers with outside option $r \in [0, r^*]$ accept offers from both firms. Workers with outside option $r \in (r^*, \max\{w_1, w_2\}]$ accept offers from firm 1 only. The rest of the workers accept offers from neither of the firms.

The steady-state subgame equilibrium constructed above, $\Sigma_{[\mathbf{P}]}$, is the unique steady-state non-trivial subgame equilibrium, if one exists under $\mathbf{P}$. Otherwise, it will be one of the at most two trivial steady-state subgame equilibria.

Lemma E.11. *The subgame equilibrium found by ascending process A^*, namely $\Sigma_{[\mathbf{P}]} = (\mathbf{P}, \mathbf{A})$, serves the highest rate of customers among all possible subgame equilibria induced by $\mathbf{P}$. Furthermore, if there exists a non-trivial subgame equilibrium under $\mathbf{P}$, then $\Sigma_{[\mathbf{P}]}$ is that equilibrium.*

Proof. First, we prove the second part. Suppose a non-trivial subgame equilibrium exists, namely $\Sigma' = (\mathbf{P}, \mathbf{A}')$. We will show that $\Sigma' = \Sigma_{[\mathbf{P}]}$. Consider the point in the ascending process when $\hat{k} = k'$. At this point, we must have

$$b_f(\hat{k}) = b'_f, \quad \forall f \in \mathcal{F},$$

where $b'_f = p_f + c(mw_f - k')$. Therefore, we must also have

$$D(b_1(\hat{k}), b_2(\hat{k})) = k'_1,$$
$$D(b_2(\hat{k}), b_1(\hat{k})) = k'_2.$$

This implies that the ascending process must stop at $\hat{k} = k'$, i.e. (E.3) is satisfied when $\hat{k} = k'$. Thereby, the ascending process finds a non-trivial subgame equilibrium. Proposition 6.2 then would imply that $\Sigma_{[\mathbf{P}]} = \Sigma'$.

The proof for the first part is done for two separate cases: either $\Sigma_{[\mathbf{P}]}$ is a non-trivial subgame equilibrium or not.

Case 1. Suppose that $\Sigma_{[\mathbf{P}]}$ is a non-trivial subgame equilibrium. The proof is by contradiction: consider a trivial subgame equilibrium, namely $\Sigma' = (\mathbf{P}, \mathbf{A}')$, for which $k' \geq k$. Without loss of generality suppose that $k_2' = 0$ in Σ'. This also implies that $k_1' = k'$. Observe that $b_1' = p_1 + c(mw_1 - k')$. Therefore, we must have

$$b_1' = p_1 + c(mw_1 - k') \geq p_1 + c(mw_1 - k) = b_1.$$

But $b_1' \geq b_1$ implies that $k' < k$. Contradiction.

Case 2. Suppose that $\Sigma_{[\mathbf{P}]}$ is a trivial subgame equilibrium. Without loss of generality, suppose that $k_2 = 0$. There is at most one other subgame equilibrium, namely $\Sigma' = (\mathbf{P}, \mathbf{A}')$, for which $k_1' = 0$. For the sake of contradiction, suppose that $\Sigma \neq \Sigma'$ and $k \leq k'$. Without loss of generality suppose that $k_1' = 0$ in Σ'. Observe that $k_2' = k'$ and $k_1 = k$. By the definition of the ascending process that finds $\Sigma_{[\mathbf{P}]}$, we must have

$$b_2 = p_2 + c((mw_2 - k)_+) \geq 1, \tag{E.4}$$

where the notation $(x)_+$ denotes the positive part of x. On the other hand, note that

$$b_2' = p_2 + c(mw_2 - k') \geq p_2 + c((mw_2 - k)_+) = b_2 \geq 1,$$

where the last inequality holds by (E.4). We just showed that $b_2' \geq 1$. Therefore, we must have $k' = k_2' = 0$. Consequently, $\Sigma = \Sigma'$. Contradiction. □

Proof of Fact 6.5. First, we show that Σ is non-trivial unless both firms serve 0 customers. For contradiction, suppose that firm 1 serves a positive rate of customers while firm 2 serves 0 customers. Observe that firm 2 gains a positive profit if it slightly increases wage, which gives a contradiction. Therefore, for the rest of the proof we can assume that Σ is non-trivial.

Let k_f denote the steady-state rate of customers who join firm f in Σ, and let $k = k_1 + k_2$. Also, let p, w respectively denote the price and wage at both firms. Define $b_f = p + c(mw - k)$, and observe that $b_1 = b_2$. Define $b = b_1$. Now, observe that $k_1 = D(b, b)$ and $k_2 = D(b, b)$ (which holds because all workers accept offers from both firms in Σ). □

E.4 Other selection rules

The analysis that we presented for Theorem 6.6 was under a specific tie-breaking used by workers: in case the maximum payoff for a worker is attained by multiple actions, the worker chooses the action with the smallest size. We call this the *Minimum* tie-breaking rule. In here, we consider a different tie-breaking rule, which we call the *Maximum* tie-breaking rule. We will show that under the Maximum tie-breaking rule, any payment profile always induces a unique subgame equilibrium. (Lemma E.14) Therefore, under this tie-breaking rule, the selection rule would choose the only existing option, the unique subgame equilibrium.

E.4.1 The Maximum tie-breaking rule

The Maximum tie-breaking rule is defined as follows: where there are multiple actions under which the worker attains her maximum payoff, the worker first removes all actions that contain wage offers below her outside option. Then, she chooses the action with the largest size that offers her the maximum payoff.

We will show that under the Maximum tie-breaking rule, any payment profile always induces a unique subgame equilibrium. This is proved in the in the following lemmas.

Lemma E.12. *Suppose the Maximum tie-breaking rule is used by workers. Then, if a payment profile* $\mathbf{P}$ *induces a non-trivial subgame equilibrium* $\Sigma = (\mathbf{P}, \mathbf{A})$*, it will induce no trivial subgame equilibrium* $\Sigma' = (\mathbf{P}, \mathbf{A}')$ *.*

Proof. The proof is by contradiction. Suppose Σ' exists. Without loss of generality, suppose $w'_1 \geq w'_2$.

The proof is done in two cases: either $k'_1 = 0$, or $k'_2 = 0$.

Case 1 Suppose $k'_1 = 0$. We consider two cases: (i) $k \leq k'$ and (ii) $k > k'$. First, suppose $k \leq k'$ (Recall Proposition E.6) Firm 1 offers an aggregate cost of $b'_1 = p_1 + c(mw_1 - k')$ to customers in Σ' and an aggregate cost of $b_1 = p_1 + c(mw_1 - k)$ to customers in Σ. Note that $b_1 \leq b'_1$ must hold because $k \leq k'$. Since 1 does not offer a higher aggregate cost in Σ, and since $k_2 > 0$ in Σ, then $k > k'$. Contradiction. So, case (ii) holds, i.e. $k > k'$. This implies that $b_1 > b'_1$ and $b_2 > b'_2$. (Note that firm 2 offers an aggregate cost $b'_2 = p_2 + c(mw_2 - k')$ to customers in Σ', but no customers join this firm.) This implies that $k \leq k'$. Contradiction.

Case 2 Suppose $k'_2 = 0$. In this case, Firm 2 offers an aggregate cost of $b'_2 = p_2 + c(mw'_2 - k'\frac{w'_2}{w'_1})$ to customers in Σ'. (The reason is that mw'_2 workers would accept offers from firm 2 under the Maximum tie-breaking rule, $k'\frac{w'_2}{w'_1}$ of whom will be busy working at firm 1)

We consider two cases for the rest of the proof: (i) $k \leq k'$ and (ii) $k > k'$. First, suppose case (i) holds. Then, firm 1 offers lower aggregate cost in Σ, which implies $k > k'$.

Contradiction. So, suppose case (ii) holds. Then, both firms offer higher aggregate costs in Σ than in Σ'. This implies that $k < k'$. Contradiction. □

Lemma E.13. *Suppose the Maximum tie-breaking rule is used by workers. Then, a payment profile* $\mathbf{P}$ *induces at most one trivial subgame equilibrium.*

Proof. The proof is similar to the proof of Lemma E.12. □

Lemma E.14. *Suppose the Maximum tie-breaking rule is used by workers. Then, a payment profile* $\mathbf{P}$ *induces a unique subgame equilibrium.*

Proof. By Lemma E.12, if a payment profile induces a non-trivial subgame equilibrium, it will induce no trivial subgame equilibrium. On the other hand, Proposition 6.2 implies that any payment profile induces at most one non-trivial subgame equilibrium. Therefore, if a payment profile induces a non-trivial subgame equilibrium, it will induce no other subgame equilibrium. So, suppose a payment profile induces no non-trivial subgame equilibrium. Then, it should induce at most one trivial subgame equilibrium, by Lemma E.13. Observing that such a payment profile indeed induces a trivial subgame equilibrium (which could be the $\varnothing$ subgame equilibrium) completes the proof. □

E.4.2 (Re-)Defining the firms' actions as allocation choices

A classic way to handle failure to launch problems in the two-sided platforms literature is redefining the actions of the platforms as quantity choices. We can repeat the same exercise here by defining the firm f's action as choosing the parameter k_f (the rate of customers who join the firm). After the vector (k_1, k_2) is fixed, then each firm f chooses price and wage to maximize profit subject to serving precisely k_f customers. We can show that, given (k_1, k_2), there is a unique tuple (p_f, w_f) for each firm f that maximizes the firm's profit subject to choosing price and wage. (We emphasize that the choices of p_f, w_f would not depend of the choices of p_{-f}, w_{-f}, so long as firm $-f$ is committed to serving k_{-f} customers.)

After redefining the firms' choices as above, we can compare the monopoly and duopoly equilibria as before. The result is that in thin markets, the firm chooses to serve a smaller rate of customers under the duopoly equilibrium (and with a higher price). Given the definition of G, this also implies that the customers' average welfare is lower under the duopoly equilibrium when the market is thin.

F Proof of Theorem 6.6

The proof involves several steps. It helps to go over a brief proof sketch before presenting the formal proof. We start by stating a system of (sufficient) conditions than any duopoly equilibrium must satisfy. (Subsection F.1) In fact, these conditions are the sufficient and necessary conditions for a *local* duopoly equilibrium , i.e. a duopoly equilibrium in which no firm has incentive to deviate from its payment profile to a another "nearby" payment profile. We then show that these conditions have a unique solution in the interval $(\underline{m}, \hat{m})$. The system of equations that characterize the local duopoly equilibrium has a closed form solution; we use the closed-form solution to prove the claim of the theorem for the local duopoly equilibrium. (Subsection F.2) In the last step of the proof, we show that this unique solution in fact represents a (global) duopoly equilibrium. (Subsection F.3)

F.1 Local duopoly equilibrium

Definition F.1. A non-trivial subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is called a *local* duopoly equilibrium if there exists an open ball $B \subset \mathbb{R}^2 \times \mathbb{R}^2$ around $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ such that for any firm f and any $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$ with $\overline{\mathbf{P}} \in B$, we have $\Pi_f(\Sigma_{[\mathbf{P}]}) \geq \Pi_f(\Sigma_{[\overline{\mathbf{P}}]})$.

Fact F.2. In Definition F.1, when given a non-trivial subgame equilibrium Σ, one can always choose the ball B such that any $\overline{\mathbf{P}} \in B$ induces a non-trivial subgame equilibrium, which will also be unique by Proposition 6.2. Therefore, $\Sigma_{[\overline{\mathbf{P}}]}$ would be the unique non-trivial subgame equilibrium under $\overline{\mathbf{P}}$.

We use Fact F.2 to write the conditions that characterize a local duopoly equilibrium. First, we need to define a notation. Let $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ be defined as follows: for any payment profile $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$, $D(\mathbf{P}_f, \mathbf{P}_{-f})$ is the steady-state rate of customers who join firm f in $\Sigma_{[\mathbf{P}]}$.[23] For notational simplicity, we sometimes use $D(p_1, w_1; p_2, w_2)$ to denote $D(\mathbf{P}_1, \mathbf{P}_2)$, where $\mathbf{P}_1 = (p_1, w_1)$ and $\mathbf{P}_2 = (p_2, w_2)$. Also, we use the notations $D_1(p_1, w_1; p_2, w_2)$ and $D_2(p_1, w_1; p_2, w_2)$ respectively to denote the derivatives of D with respect to its first and second argument, evaluated at the payment profile $(p_1, w_1; p_2, w_2)$

We are now ready to state the conditions which characterize a local duopoly equilibrium.

[23] Note that we are using the symmetry of the customers' demand for the two firms in writing such a functional form for determining the rate of customers who join a firm.

Proposition F.3. *A subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ is a local duopoly equilibrium if and only if it satisfies*

$$k_1 = -D_1(p_1, w_1; p_2, w_2) \cdot (p_1 - w_1), \tag{F.1}$$
$$k_1 = D_2(p_1, w_1; p_2, w_2) \cdot (p_1 - w_1), \tag{F.2}$$
$$k_1 = D(p_1, w_1; p_2, w_2), \tag{F.3}$$
$$k_2 = -D_1(p_2, w_2; p_1, w_1) \cdot (p_2 - w_2), \tag{F.4}$$
$$k_2 = D_2(p_2, w_2; p_1, w_1) \cdot (p_2 - w_2), \tag{F.5}$$
$$k_2 = D(p_2, w_2; p_1, w_1). \tag{F.6}$$

Proof. First, we show that any local duopoly equilibrium must satisfy the given conditions. (F.1) and (F.2) are the FOCs of firm 1 for price and wage, respectively. (F.3) is a balance equation: on the LHS we have the rate of customers who join the firm, and on the RHS we have the rate of customers who gain positive payoff from joining the firm. This could also be interpreted as a market clearing condition. Equations (F.4),(F.5), and (F.6) are the same equations but written for firm 2.

Next, we show that if the given equations are satisfied for a given Σ, then it is a local duopoly equilibrium. Observe that the FOCs for firm f imply that $\nabla \Pi_f(p_f, w_f) = 0$. This guarantees the existence of a ball B around $\mathbf{P}$ that satisfies the condition given in Definition F.1. □

Corollary F.4 (of Proposition F.3)**.** *If the tuples (p_1, w_1, k_1) and (p_2, w_2, k_2) with $k_1, k_2 > 0$ satisfy the conditions given in Proposition F.3, then the payment profile $\mathbf{P} = ((p_1, w_1), (p_2, w_2))$ induces a non-trivial subgame equilibrium Σ which is also a local duopoly equilibrium.*

Proof. The proof is similar to the proof of the second part of Proposition F.3; in addition to that, observe that conditions (F.3) and (F.6) guarantee that the payment profile $\mathbf{P}$ induces a non-trivial subgame equilibrium. □

Proposition F.5. *A subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ with $\mathbf{P}_f = (p, w)$ for $f \in \mathcal{F}$ and $\mathbf{k} = (k_1, k_2) = (k/2, k/2)$ is a symmetric local duopoly equilibrium if and only if it satisfies*

$$k/2 = -D_1(p, w; p, w) \cdot (p - w), \tag{F.7}$$
$$c'(mw - k) = \frac{-1}{m}, \tag{F.8}$$
$$k/2 = D(p, w; p, w). \tag{F.9}$$

Proof. First, we show that a local duopoly equilibrium satisfies the given conditions. (F.7) is the FOC for price, which is identical for both firms by symmetry. (F.9) is the balance equation, which is identical for both firms. On its LHS we have the rate of customers who join the firm, and on its RHS we have the rate of customers who gain positive payoff from joining the firm. This condition could also be interpreted as a market clearing condition. To derive (F.8), observe that if (F.8) does not hold, then the firm could increase its profit by changing price and wage: if $c'(mw_1 - k) > \frac{-1}{m}$, then firm 1 could increase its profit by decreasing wage by some sufficiently small $\epsilon > 0$ and decreasing price by a positive $\epsilon' < \epsilon$, while doing this so that the aggregate cost offered to its customers does not change. Hence, the rate of the customers that the firm serves would not change, but the firm's commission fee would go up, and therefore, the firm's profit. If $c'(mw_1 - k) < \frac{-1}{m}$, then the same could be done, but by increasing wage and price. Condition (F.8) could also be derived in another way, similar to what we showed in Proposition B.3 for the case of monopoly: the FOC of a firm with respect to wage is

$$k/2 = D_2(p, w; p, w) \cdot (p - w). \tag{F.10}$$

Equating the RHS of the above equation with the RHS of (F.7) implies (F.8). This is shown in files "duo-focp" and "duo-focw", which compute the FOCs with respect to price and wage. The RHS of the FOCs should be equal (because their left-hand sides are equal); equating the right-hand sides implies (F.8).

Next, we show that if a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ satisfies the above conditions, then it should be a local duopoly equilibrium. To this end, we prove that (F.7) and (F.8) together imply that the FOC of a firm f with respect to wage holds. This is is derived from the same exercise that we did before: we saw that equating the right-hand sides of (F.7) and (F.10) implies (F.8). Similarly, we show that (F.7) and (F.8) together imply (F.10). This is shown in file "retrieving-focw ". □

Corollary F.6 (of Proposition F.5)**.** *If the tuple (p, w, k) with $k > 0$ satisfies the conditions given in Proposition F.5, then the payment profile $\mathbf{P} = ((p, w), (p, w))$ induces a non-trivial subgame equilibrium Σ which is also a local duopoly equilibrium.*

Proof. The proof is similar to the proof of the second part of Proposition F.5; in addition to that, observe that (F.9) guarantees that the payment profile $\mathbf{P}$ induces a non-trivial subgame equilibrium. □

We also need to define some notation for the notion of monopoly equilibrium. We define the monopoly equilibrium in our setting by removing firm 2 and letting firm 1 to choose its optimal price and wage; the definition of the monopoly equilibrium the same as in Section 3. To help readability, we repeat the conditions that characterize the monopoly equilibrium, using the demand function D. We let $D(p, w; \infty, 0)$ denote the rate of customers who join firm 1 when its payment profile is (p, w) and firm 2 is removed. Note that removing firm 2 is equivalent to defining the payment profile for firm 2 to be $(\infty, 0)$.

Proposition F.7. *A tuple (p, w, k) defines a monopoly equilibrium under firm 1 with p, w being the price and wage offered by firm 1 and k being the rate of customers who join the firm, if and only if the tuple satisfies the following conditions:*

$$k = -D_1(p, w; \infty, 0) \cdot (p - w), \tag{F.11}$$

$$c'(mw - k) = \frac{-1}{m}, \tag{F.12}$$

$$k = D(p, w; \infty, 0). \tag{F.13}$$

Proof. The proof is followed from our analysis of the monopoly equilibrium in Section 4. □

Definition F.8. *Given m, let $\mathsf{DE}(m)$ denote the system of three equations given in Proposition F.5, and let $\mathsf{ME}(m)$ denote the system of three equations given in Proposition F.7.*

Lemma F.9. *Let $m_0 = \frac{1}{-c'(0)}$. When $m \leq m_0$, there is no tuple (p, w, k) with $k > 0$ that satisfies $\mathsf{DE}(m)$ or $\mathsf{ME}(m)$.*

Proof. The proof is based on equations (F.8) and (F.12). First, suppose that $m \leq m_0$ and there exists a tuple (p, w, k) with $k > 0$ that satisfies $\mathsf{DE}(m)$. Define $i = mw - k$. Note that $i \geq 0$ must hold, otherwise the (p, w, k) is not a valid solution, since the argument of c falls outside of its domain. Furthermore, $i > 0$ should hold, otherwise $k > 0$ cannot hold because c is a standard cost function.

Because c is convex, its derivative is increasing. Therefore, $c'(i) > c'(0)$ must always hold, which implies that $\frac{-1}{m} > c'(0)$. Consequently, $m > -\frac{1}{c'(0)}$ holds, which proves the claim for the system $\mathsf{DE}(m)$. The proof for the system $\mathsf{ME}(m)$ is identical. □

F.2 Proving the theorem's claim for local equilibria

We set $\underline{m} = m_0$. Recall from Lemma F.9 recall that there is no monopoly or duopoly equilibrium when $m \leq \hat{m}$. In this step of the proof, we show the existence of $\hat{m} > m_0$ such that for any $m \in (\underline{m}, \hat{m})$, the following holds: (i) there is unique solution to $\mathsf{DE}(m)$, which we denote by the tuple $(p_{\text{duo}}(m), w_{\text{duo}}(m), k_{\text{duo}}(m))$, (ii) there is unique solution to $\mathsf{ME}(m)$, which we denote by the tuple $(p_{\text{mon}}(m), w_{\text{mon}}(m), k_{\text{mon}}(m))$, and (iii) $p_{\text{duo}}(m) > p_{\text{mon}}(m)$ and $u^C_{\text{duo}}(m) < u^C_{\text{mon}}(m)$.

We do not set the value for $\hat{m}$ now; its existence would be proved by the end of the proof.

Lemma F.10. *There exists m' such that for all $m \in (\underline{m}, m')$, there exists a unique monopoly equilibrium at m.*

Proof. First, observe that for any $m > \underline{m}$, there exists at least one monopoly equilibrium at m. This is implied by Lemma C.1. To prove the existence of m' and proving the uniqueness of equilibrium in the interval $(\underline{m}, m')$, we just find the closed-form expressions for all the (possibly complex) roots of $\mathsf{ME}(m)$. (See file "ME-sols") This system reduces to an equation of degree 3 in p, and therefore, there are 3 (possibly complex) solutions to the system . We retrieve the three solutions. Let $k^1(m), k^2(m), k^3(m)$ denote the values that each of these solutions assign to the variable k, as a function of m. We observe that $k^1(m) \equiv 0$, and $\lim_{m \to m_0} k^2(m) > 0$; this rules out two of the solutions in an interval $(\underline{m}, m')$, when m' is chosen to be sufficiently close to m. Therefore, the exists at most one monopoly equilibrium in the interval $(\underline{m}, m')$. This result, together with the first part of the proof prove the claim. □

Lemma F.11. *There exists m' such that for all $m \in (\underline{m}, m')$, there exists a unique symmetric local duopoly equilibrium at m.*

Proof. First, we need to recall a notation from Subsection E.1: $D : [0,1]^2 \to [0,1]$ is the demand function of customers when in terms of the aggregate costs that the firms offer, i.e. $D(b_1, b_2)$ determines the mass of customers demanding to join firm 1 assuming that the aggregate cost at firm f is b_f.

We do not set the value of m' immediately, its value will be set during the proof. Fix m, and consider a continuum of subgame equilibria, $\{\Sigma^\beta : \beta \in [0, \bar{\beta}]\}$, defined as follows: $\bar{\beta} < 1$ is a positive constant which we will define later, at the same time that we define m'. Σ^β is a subgame equilibrium defined as follows. The variable $b^\beta \equiv 1 - \beta$ represents the aggregate cost that is offered to customers in Σ^β. Therefore, Σ^β is just the $\varnothing$ subgame equilibrium for $\beta = 0$. For positive β, $\Sigma^\beta = (\mathbf{P}^\beta, \mathbf{A}^\beta)$ is defined to be the symmetric subgame equilibrium with customer composition

$$\mathbf{k}^\beta = (k^\beta_1, k^\beta_2) = (D(b^\beta, b^\beta), D(b^\beta, b^\beta))$$

and payment profile $\mathbf{P}^\beta = ((p^\beta, w^\beta), (p^\beta, w^\beta))$ such that

$$p^\beta + c(mw^\beta - k^\beta) = b^\beta, \tag{F.14}$$

$$c'(mw^\beta - k^\beta) = \frac{-1}{m}. \tag{F.15}$$

where $k^\beta = k_1^\beta + k_2^\beta$ (as in our conventional notation). It is straight-forward to see that given any fixed b^β, there is a unique pair (p^β, w^β) that satisfies (F.14) and (F.15).

Define $\mathfrak{L}(\beta)$ to be

$$\mathfrak{L}(\beta) = \left(-\frac{D_1(p^\beta, w^\beta; p^\beta, w^\beta)}{D(p^\beta, w^\beta; p^\beta, w^\beta))}\right) \cdot (p^\beta - w^\beta). \tag{F.16}$$

The significance of this definition is that when Σ^β is a local duopoly equilibrium, we must have $\mathfrak{L}(\beta) = 1$: only then (F.16) will coincide with the firm's FOC for price, as given in (F.1).

In the rest of the proof, we will show that (i) $\lim_{\beta\to 0} \mathfrak{L}(\beta) = \infty$, (ii) $\lim_{\beta\to\bar{\beta}} \mathfrak{L}(\beta) = 0$ for a positive $\bar{\beta}$ which will be defined, and (iii) $\mathfrak{L}(\beta)$ is continuous at any $\beta \in (0, \bar{\beta})$. Proving these three steps completes the proof because they imply that $\mathfrak{L}(\beta^*) = 1$ for some $\beta^* \in (0, \bar{\beta})$. It is then straight-forward to verify that Σ^{β^*} satisfies all the conditions in Proposition F.5, and therefore, it is a symmetric local duopoly equilibrium.

Proof for step (i) There are two multiplicands on the RHS of (F.16). We denote the first one by $f(\beta)$ and the second one by $g(\beta)$. We will prove that $\lim_{\beta\to 0} f(\beta) = \infty$, and $\lim_{\beta\to 0} g(\beta) > 0$. This would complete this step.

In file "el-case1", we derive the closed-form expression for $f(\beta)$:

$$f(\beta) = \frac{2\left(a^2 m - am + b^\beta - 1\right)}{(1 - b^\beta)\left(a^2 m - am + 2b^\beta - 2\right)},$$

where $a = 1 - \sigma$. This implies $\lim_{\beta\to 0} \mathfrak{L}(\beta) = \infty$.

To finish this step, it remains to prove $\lim_{\beta\to 0} g(\beta) > 0$. Because $m > \underline{m} = \frac{1}{-c'(0)}$, then there exists a non-trivial subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ at m. (The proof is essentially identical to the proof of Lemma C.1) Moreover, we choose Σ such that it satisfies the equation $c'(mw_1 - k) = \frac{-1}{m}$.

Claim F.12. *Σ could be chosen such that it satisfies $c'(mw_1 - k) = \frac{-1}{m}$.*

Proof. The proof is similar to the proof that we gave for (F.8) in Proposition F.5. For completeness, we briefly repeat that argument here.

First, recall the expression that Proposition E.6 gives for the waiting cost of customers in any non-trivial subgame equilibrium: the waiting cost at firm f in Σ is then equal to $c(mw_f - k)$. Suppose $c'(mw_1 - k) = \frac{-1}{m}$ does not hold. We then will show that firm 1 could choose another payment profile which increase its profit but does not change the rate of customers that are served by firm 1 or firm 2. First, suppose that $c'(mw_1 - k) > \frac{-1}{m}$. Then, firm 1 could increase its profit by decreasing wage by some sufficiently small $\epsilon > 0$ and decreasing price by a positive $\epsilon' < \epsilon$, while doing this so that the aggregate cost offered to its customers does not change. Hence, the rate of the customers that firm 1 serves would not change, but the firm's commission fee would go up, and therefore, the firm's profit. Let the new payment profile of firm 1 be denoted by $\mathbf{P}'_1$, and let $\mathbf{P}' = (\mathbf{P}'_1, \mathbf{P}_2)$. Observe that the payment profile $\mathbf{P}'$ induces a subgame equilibrium Σ' which has exactly the same customer composition as Σ. Therefore, Σ' is non-trivial; this proves the claim.

If $c'(mw_1 - k) < \frac{-1}{m}$, then the same argument applies, but by increasing wage and price, instead of decreasing them as above. □

Let b_f denote the aggregate cost of firm f in Σ. Observe that for any β, customers incur the same waiting cost in Σ^β and Σ, because $c'(mw_1 - k) = c'(mw_1^\beta - k^\beta) = \frac{-1}{m}$. However, note that $b_1 < b_1^\beta$ holds for β sufficiently close to 0. The two latter facts together imply that that $p_1 < p^\beta$ holds for β sufficiently close to 0.

We also prove that $w_1 > w^\beta$ holds for β sufficiently close to 0: because $c'(mw_1 - k) = c'(mw_1^\beta - k^\beta) = \frac{-1}{m}$, and because $k^\beta < k$ for β sufficiently close to 0.

We have shown that $p_1 < p^\beta$ and $w_1 > w^\beta$ hold for β sufficiently close to 0. Therefore, we must have

$$\lim_{\beta \to 0} p^\beta - w^\beta > p_1 - w_1.$$

Consequently, $\lim_{\beta \to 0} g(\beta) > 0$, and this step is complete.

Proof for step (ii) Consider the family of all symmetric subgame equilibrium with payment profiles $((p, w), (p, w))$ such that $p = w$. Let $\underline{b}$ denote the infimum of the aggregate cost that customers incur in this family. Define $\overline{\beta} = 1 - \underline{b}$. It is straight-forward to verify that $\lim_{\beta \to \overline{\beta}} g(\beta) = 0$, i.e. as β approaches $\overline{\beta}$, the commission fee in Σ^β approaches 0.

To complete this step, it remains to show that $\lim_{\beta \to \overline{\beta}} f(\beta)$ exists and is finite. Recall that

$$f(\beta) = \left(-\frac{D_1(p^\beta, w^\beta; p^\beta, w^\beta)}{D(p^\beta, w^\beta; p^\beta, w^\beta))}\right).$$

To ensure existence of the limit, we prove that $f(\beta)$ is continuous. To ensure continuouity, we choose m' sufficiently close to $\underline{m}$ so that the $\overline{\beta} < \frac{1-\sigma}{2}$. Under this assumption, the closed-form expression for $f(\beta)$ is computed in file "el-case1". This assumption ensures that the derivative on the RHS of the above equation exists for all $\beta \leq \overline{\beta}$, and therefore ensures the continuity of $f(\beta)$ for all such β. (We note that the continuity is ensured for much larger m', as the derivative exists for all $\beta < \sigma$, and possibly for larger β; but we make the stronger assumption to keep the proof simple.)

We have established that $\lim_{\beta\to\overline{\beta}} g(\beta) = 0$, and that $\lim_{\beta\to\overline{\beta}} f(\beta)$ exists and is finite. This completes step (ii).

Proof for step (iii) To complete this step, we need to show that $f(\beta)$ and $g(\beta)$ are continuous in the interval $(0, \bar{\beta})$. We proved the claim for $f(\beta)$ in step (ii). It remains to prove the claim for $g(\beta)$. Recall that $g(\beta) = p^\beta - w^\beta$, and observe that

$$p^\beta = \beta - c(c'^{-1}(-1/m)),$$
$$w^\beta = \frac{c'^{-1}(-1/m) + D(b^\beta, b^\beta)}{m},$$

which are continuous functions in β.

□

Lemma F.13. *There exists m' such that for all $m \in (\underline{m}, m')$, a unique monopoly equilibrium and a unique symmetric duopoly equilibrium exist at m, and furthermore, $p_{duo}(m) > p_{mon}(m)$ and $u^C_{duo}(m) < u^C_{mon}(m)$.*

Proof. The existence and uniqueness of the equilibria is implied by Lemma F.10 and Lemma F.11. It remains to prove that $p_{\mathsf{duo}}(m) > p_{\mathsf{mon}}(m)$ and $u^C_{\mathsf{duo}}(m) < u^C_{\mathsf{mon}}(m)$ hold.

To prove the theorem, we extend the domain of functions $p_{\mathsf{duo}}(m), w_{\mathsf{duo}}(m), k_{\mathsf{duo}}(m)$ and $p_{\mathsf{mon}}(m), w_{\mathsf{mon}}(m), k_{\mathsf{mon}}(m)$ to $[\underline{m}, \infty)$, so that their domain includes the point $\underline{m}$, and define the value of all these functions to be 0 at point $\underline{m}$. In files "DE-sols" and "ME-sols" we show that the limit of all of these six functions is 0 as m approaches m_0. So, this extension is just the natural continuous extension.

First, we prove $p_{\mathsf{duo}}(m) > p_{\mathsf{mon}}(m)$. To this end, we define a function $d(m) = p_{\mathsf{duo}}(m) - p_{\mathsf{mon}}(m)$, and apply a derivative test on it at point m_0. We will show that $d^{(i)}(m_0) = 0$

for all positive integers $i \leq 3$ and $d^{(4)}(m_0) > 0$,where the notation $d^{(i)}(m_0)$ denotes the i-th derivative of d with respect to m computed at m_0. This derivative test implies that the function $d(m)$ is increasing at m_0, i.e. there exists m' such that d is increasing over the interval $(\underline{m}, m')$. Since $d(0) = 0$ by definition, then $p_{\text{duo}}(m) > p_{\text{mon}}(m)$ must hold when $m \in (\underline{m}, m')$. The claim about the derivatives $d^{(i)}$ is proved in file "d". There, we compute the derivatives directly from the closed-form solutions for $p_{\text{mon}}(m)$ and $p_{\text{duo}}(m)$, which are computed in files "ME-sols' and "DE-sols", respectively. □

F.3 Proving that local equilibria are global equilibria

In the previous step of the proof (Subsection F.2) we showed that for any $m \in (\underline{m}, m')$, there exists a unique symmetric local duopoly equilibrium at m. In this step, the last step of the proof, we show that there exists $m'' \in (\underline{m}, m')$ such that any symmetric local equilibrium in the interval $(\underline{m}, m')$ is also a (global) duopoly equilibrium. Setting $\hat{m} = m''$ will then prove the claim of the theorem.[24] We do not fix the value of m'' in advance; this value will be set in the course of the proof.

We need one definition before presenting the proof. We say that a firm f has a *standard* payment profile in the subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$ if the condition

$$c'(mw_f - k) = \frac{-1}{m}$$

is satisfied. Intuitively, this conditon means that holding k_f fixed, firm f has chosen price and wage optimally. (Recall the proof of Claim F.12 where we saw that if $c'(mw_f - k) \neq \frac{-1}{m}$, then firm f can choose a different price and wage to increase its profit, while serving the same rate of customers as before.)

Given a subgame equilibrium $\Sigma = (\mathbf{P}, \mathbf{A})$, we say that a deviation $\overline{\mathbf{P}}_f$ is a *standard deviation* for firm f if $\overline{\mathbf{P}}_f$ is a standard payment profile for firm f in $\Sigma_{[\overline{P}]}$, where $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$. This definition is important in proving that any local duopoly equilibrium is also a (global) duopoly equilibrium: to prove this, without loss of generality, it sufficies to prove that the standard deviations for firm f cannot increase its profit. More precisely, let $\overline{\mathbf{P}} = (\overline{\mathbf{P}}_f, \mathbf{P}_{-f})$ denote the payment profile after the deviation of firm f. The idea is that if $\overline{\mathbf{P}}_f$ is not a standard deviation for f, then firm f can choose a standard deviation $\overline{\overline{\mathbf{P}}}_f$ which increases her profit more than the deviation $\overline{P}_f$. This claim holds essentially by the

[24] We believe that any symmetric local duopoly equilibrium is also a (global) duopoly equilibrium, however, proving the claim for the interval $(\underline{m}, m'')$ seems to be significantly simpler.

same argument that proves Claim F.12. This significantly simplifies the analysis, as it helps to interpret the game played between the firms as a game in which the firms compete by offering aggregate costs to customers, rather than offering a price to customers and a wage to workers. The details will become clear in the course of this proof.

Fix m, and let the symmetric local duopoly equilibrium at m be denoted by $\Sigma = (\mathbf{P}, \mathbf{A})$, with $\mathbf{P} = ((p_1, w_1), (p_2, w_2))$ and customer composition $\mathbf{k} = (k_1, k_2)$. The duopoly equilibrium is symmetric, and hence we let $p = p_1 = p_2$, $w = w_1 = w_2$, and $k_1 = k_2 = k/2$. Therefore, the total rate of customers that join a firm in Σ is k. As usual, we use the variable k_f to denote the rate of customers who join firm f in Σ; so, $k_f = k/2$. As usual, we let b_f denote the aggregate cost offered to customers at firm f.

Suppose that firm 1 makes a deviation after which the aggregate cost that it offers to customers changes $b_1^{\#} = b_1 - \delta$. We prove that such a deviation does not increase the profit of firm 1, for any positive or negative δ. We can assume that the deviation made by firm 1 is a standard deviation; this is without loss of generality: if the deviation is not standard, then firm 1 can choose a new deviation which is standard and increases her profit more than the non-standard deviation. (This follows from the same argument in the proof of Claim F.12).

We present the proof for the case of $\delta > 0$ first. The case of $\delta < 0$ is very similar, and will be proved at the end. Let $p_1^{\#}, w_1^{\#}, k_1^{\#}$ respectively denote the price and wage offered by firm 1, and the mass of customers that join firm 1 after her deviation. Also, let $k_2^{\#}$ denote the total mass of customers who join firm 2 after firm 1's deviation, and let $k^{\#} = k_1^{\#} + k_2^{\#}$.

Next, we will show that all the variables $p_1^{\#}, w_1^{\#}, k_1^{\#}, k_2^{\#}, k^{\#}$ are uniquely determined once δ is fixed. (Later in the proof, we will use this property to write these variables as a function of δ.) It is straight-forward to compute $k_1^{\#}$ as a function of $b_1^{\#}$: by our choice of m', $k_1^{\#}$ is the area of the shaded triangle in Figure 12, which could clearly be written as a function of $b_1^{\#}$. We use the market-clearing equation to show why $k_2^{\#}$ is uniquely determined by δ:

$$k_1^{\#} + \hat{k}_2 = k_1^{\#} + D(p_2 + c(mw_2 - k_1^{\#} - \hat{k}_2), b_1^{\#}).$$

In the above equation, we have used the variable $\hat{k}_2$ to take the place of $k_2^{\#}$. Observe that the LHS of the above equation is strictly increasing in $\hat{k}_2$, but its RHS is decreasing in $\hat{k}_2$. There exists a unique value of $\hat{k}_2$ that solves the above equation, which we called $k_2^{\#}$. This also implies that $k^{\#}$ is uniquely determined by δ, since $k^{\#} = k_1^{\#} + k_2^{\#}$.

Because $c'(mw_1^{\#} - k^{\#}) = \frac{-1}{m}$, then $w_1^{\#}$ is also uniquely determined by δ, and so is $p_1^{\#}$, because $b_1^{\#} = p_1^{\#} + c(mw_1^{\#} - k^{\#})$. Moreover,

$$c'(mw_1^{\#} - k^{\#}) = c'(mw_1 - k) = \frac{-1}{m}$$

implies $p_1 - p_1^{\#} = \delta$.

The profit of firm 1 before and after the deviation, respectively, is

$$\Pi_1 = k_1 \cdot (p_1 - w_1),$$
$$\Pi_1^{\#} = k_1^{\#} \cdot (p_1^{\#} - w_1^{\#}).$$

Next, we write $\Pi_1^{\#}$ in a slightly different form:

$$\Pi_1^{\#} = \left(k_1 + \Delta_{[k_1]}\right) \cdot \left(p_1 - w_1 - \Delta_{[p_1]} - \Delta_{[w_1]}\right),$$

where

$$\Delta_{[k_1]} = k_1^{\#} - k_1,$$
$$\Delta_{[p_1]} = p_1 - p_1^{\#} = \delta,$$
$$\Delta_{[w_1]} = w_1^{\#} - w_1.$$

Later on, we will see that

$$\Delta_{[k_1]}, \Delta_{[p_1]}, \Delta_{[w_1]} > 0.$$

In terms of these new variables, we can now write the inequality $\Pi_1 \geq \Pi_1^{\#}$ as

$$(k_1 + \Delta_{[k_1]}) \cdot (\Delta_{[w_1]} + \Delta_{[p_1]}) \geq \Delta_{[k_1]} \cdot (p_1 - w_1). \tag{F.17}$$

We will show that the above inequality holds for all $\delta > 0$.

We start by providing a lower-bound for $\Delta_{[w_1]}$. First, observe that $w_1^{\#} = \frac{c'^{-1}(-1/m)+k^{\#}}{m}$. To provide the promised lower-bound we will show that $k^{\#}$, when written as a function of δ, is increasing and convex in δ. After proving this claim, we use the derivative of $k^{\#}$ with respect to δ at point $\delta = 0$ to compute a lower-bound for $w_1^{\#}$, which will turn into a lower-bound for $\Delta_{[w_1]}$.

To make this argument precise, we use the notation $k^{\#}(\delta)$ to denote $k^{\#}$ as a function of δ. Similarly, we define functions $b_1^{\#}(\delta), b_2^{\#}(\delta)$ to denote the values of $b_1^{\#}$ and $b_2^{\#}$ as functions of δ. Let the interval $[0, \bar{\delta}]$ denote the minimal interval that all possible values of δ belong to.

Claim F.14. *$k^{\#}(\delta)$ is increasing and convex over $[0, \bar{\delta}]$.*

Proof. In file "ksharp-increasing" we compute $\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}$ by implicit differentiation from the market-clearing equation with respect to δ. We compute

$$\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta} = -\frac{1 - b_1^{\#}(\delta)}{(a-1)a - (b_2^{\#}(\delta) - 1) \cdot c'\left(mw_2^{\#} - k^{\#}(\delta)\right)} > 0,$$

where the inequality holds because $0 \leq b_1^{\#}(\delta) < 1$ and $0 < b_2^{\#}(\delta) \leq 1$ hold for all $\delta \in [0, \bar{\delta}]$.

To prove the convexity claim, in file "ksharp-convex" we compute $\frac{\mathrm{d}^2\,k^{\#}(\delta)}{(\mathrm{d}\,\delta)^2}$. The key point is that

$$\lim_{\delta \to 0} \frac{\mathrm{d}^2\,k^{\#}(\delta)}{(\mathrm{d}\,\delta)^2} = \left.\frac{\mathrm{d}^2\,k^{\#}(\delta)}{(\mathrm{d}\,\delta)^2}\right|_{\delta=0} = \frac{1}{a - a^2} > 0, \tag{F.18}$$

holds for any $m > \underline{m}$. First of all, this implies the convexity of $k^{\#}(\delta)$ at $\delta = 0$. Second, we will show that we can choose m sufficiently close to $\underline{m}$ such that $\frac{\mathrm{d}^2\,k^{\#}(\delta)}{(\mathrm{d}\,\delta)^2} > 0$ is guaranteed to hold for all $\delta \in [0, \bar{\delta}]$. The idea is choosing m sufficiently close to $\underline{m}$ so that $\bar{\delta}$ becomes sufficiently close to 0. Convexity of $k^{\#}(\delta)$ at any $\delta \in [0, \bar{\delta}]$ would then be implied by the continuity of $\frac{\mathrm{d}^2\,k^{\#}(\delta)}{(\mathrm{d}\,\delta)^2}$ and the fact that (F.18) holds.

Choose ζ such that for any positive $\kappa < \zeta$ and any $m > \underline{m}$, we have

$$\frac{\mathrm{d}^2\,k^{\#}(\kappa)}{(\mathrm{d}\,\kappa)^2} > 0.$$

Existence of such $\zeta > 0$ is proved by investigating the closed-form expression for $\frac{\mathrm{d}^2\,k^{\#}(\kappa)}{(\mathrm{d}\,\kappa)^2}$, as done in file "ksharp-convex".

We can then choose m'' such that for any $m < m''$, $\bar{\delta}(m) < \zeta$. This proves the claim.[25] □

In the rest of the proof, we prove (F.17) by proving a stronger version of it. As we discussed earlier, this stronger version is derived by writing a lower-bound for $\Delta_{[w_1]} = \frac{k^{\#} - k}{m}$. Using Claim F.14, we can write

$$\Delta_{[w_1]} = \frac{k^{\#}(\delta) - k^{\#}(0)}{m} \geq \frac{\delta}{m} \cdot \left(\left.\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\right|_{\delta=0} \right) \tag{F.19}$$

[25]The fact that m is sufficiently small buys some technical simplicity. However, we believe that the convexity claim still holds for larger m.

We now write a stronger version of (F.17) using the above expression, while dividing both sides of (F.17) by $\Delta_{[k_1]}$:

$$\left(1+\frac{k_1}{\Delta_{[k_1]}}\right)\cdot\left(\left(\frac{\delta}{m}\cdot\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}\right)+\delta\right)\geq p_1-w_1. \tag{F.20}$$

Define $\epsilon=\frac{\delta}{1-a}$. In order to simplify (F.20), we also use the fact that

$$k_1=\frac{xy}{2}, \tag{F.21}$$

$$\Delta_{[k_1]}=\epsilon y+\frac{\epsilon^2 y}{2x}, \tag{F.22}$$

where $x=1-\frac{b_1-a}{1-a}$, $y=\frac{1-b_1}{a}$, and $a=1-\sigma$, as illustrated in Figure 12 (This figure depicts customers' valuations using the unit square, similar to Figure 11). Computing the value of $\Delta_{[k_1]}$ in terms of x, y, ϵ is straight-forward, as shown in Figure 13.

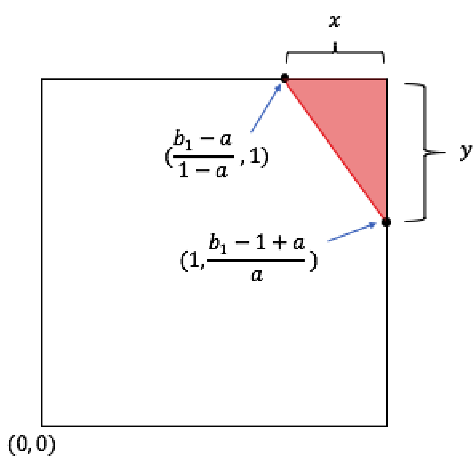


Figure 12: The shaded area represents the customers who join firm 1.

We now can rewrite (F.20) in the following way

$$\left(1+\frac{xy/2}{\epsilon y+\frac{\epsilon^2 y}{2x}}\right)\cdot\left(\left(\frac{\delta}{m}\cdot\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}\right)+\delta\right)\geq p_1-w_1,$$

which could be simplified to

$$\left(\epsilon+\frac{x}{2+\frac{\epsilon}{x}}\right)\cdot\left(\left(\frac{1-a}{m}\cdot\frac{\mathrm{d}\,k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}\right)+1-a\right)\geq p_1-w_1. \tag{F.23}$$

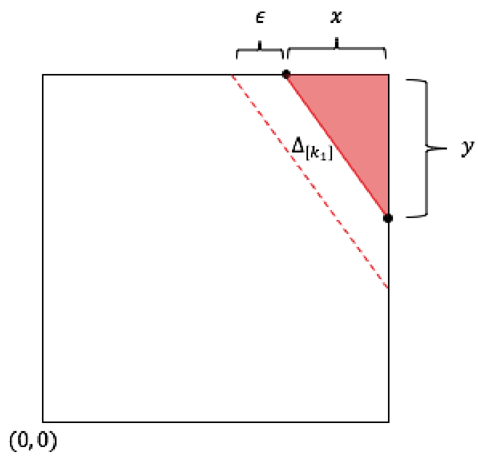


Figure 13: $\Delta_{[k_1]}$ is the area of the trapezoid, and can be written in terms of x, y, ϵ.

To prove that the above inequality holds, we observe that the the first multiplicand on the LHS is a function of ϵ, where as the second multiplicand is not. Therefore, to prove (F.23), it suffices to prove the following: (i) The first multiplicand is an increasing function of ϵ, and (ii) Equation (F.23) holds at $\epsilon = 0$. These are proved in Steps (i) and (ii), respectively.

Step (i) We compute the derivative of the first multiplicand on the LHS of (F.23)

$$\frac{\mathrm{d}\left(\epsilon + \frac{x}{2+\frac{\epsilon}{x}}\right)}{\mathrm{d}\,\epsilon} = 1 - \frac{x^2}{(\epsilon + 2x)^2} > 0,$$

which is always positive. This completes step (i).

Step (ii) First, we rewrite (F.23) as follows.

$$\left(\delta + \frac{x(1-a)}{2 + \frac{\delta}{(1-a)x}}\right) \cdot \left(\left(\frac{1}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\,\delta}\Big|_{\delta=0}\right) + 1\right) \geq p_1 - w_1. \tag{F.24}$$

To prove (F.23) for $\epsilon = 0$, we instead prove (F.24) for $\delta = 0$. Intuitively, (F.24) holds because we assumed the given subgame equilibrium is a *local* duopoly equilibrium, and therefore for sufficiently small $\delta > 0$, firm 1's deviation should not increase her profit. More precisely, we demonstrate that when $\delta = 0$, (F.24) in fact coincides with firm's FOC that ensures no (local) *standard* deviation is beneficial to the firm. To see why this holds, observe that when

$\delta = 0$, then (i) the term $\left(\delta + \frac{x(1-a)}{2+\frac{\delta}{(1-a)x}}\right)$ in (F.24) is just equal to

$$\frac{k^{\#}(\delta)}{\frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}},$$

and (ii) the term $\left(\left(\frac{1}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}\right) + 1\right)$ is equal to

$$-\frac{\mathrm{d}\, p^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0} + \frac{\mathrm{d}\, w^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}.$$

This complete the proof for the case of $\delta > 0$. It remains to address the case of $\delta < 0$. The proof for this case is almost identical. We briefly state the proof, skipping the steps that are identical to the previous proof. We use the same variables that we used in the previous proof. The variables $\Delta_{[k]}, \Delta_{[k_1]}, \Delta_{[p_1]}, \Delta_{[w_1]}$, are defined in a way that they are all positive. Also, we define $\epsilon = \frac{-\delta}{1-a}$ so that it is a positive quantity. To prove that firm 1's profit after deviation does not increase, we then have to show that

$$(k_1 - \Delta_{[k_1]}) \cdot (\Delta_{[w_1]} + \Delta_{[p_1]}) \leq \Delta_{[k_1]} \cdot (p_1 - w_1).$$

(This is the counterpart for (F.17)) Dividing both sides by $\Delta_{[k_1]}$, the above inequality can be written as

$$\left(\frac{k_1}{\Delta_{[k_1]}} - 1\right) \cdot (\Delta_{[w_1]} + \Delta_{[p_1]}) \leq p_1 - w_1.$$

Following the same proof steps as before, we can get the counterpart for inequality (F.23) as

$$\left(\frac{xy/2}{\epsilon y - \frac{\epsilon^2 y}{2x}} - 1\right) \cdot \left(\left(\frac{1-a}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}\right) + 1 - a\right) \leq p_1 - w_1,$$

or equivalently,

$$\left(\frac{x}{2 - \frac{\epsilon}{x}} - \epsilon\right) \cdot \left(\left(\frac{1-a}{m} \cdot \frac{\mathrm{d}\, k^{\#}(\delta)}{\mathrm{d}\, \delta}\Big|_{\delta=0}\right) + 1 - a\right) \leq p_1 - w_1. \qquad \text{(F.25)}$$

We prove the above inequality in two steps: (i) We show that the first multiplicand is decreasing in ϵ, and (ii) We prove the inequality for $\epsilon = 0$. For step (i), we compute the

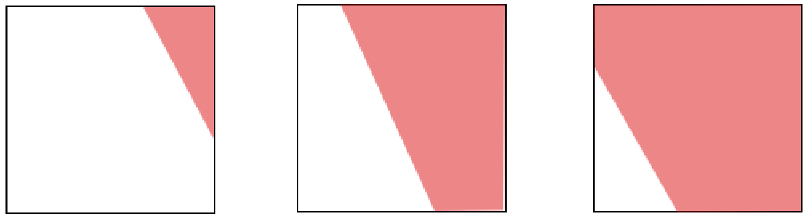


Figure 14: There are three cases for the set of customers served by a monopolist: Cases 1-3 from left to right.

derivative of the first multiplicand with respect to ϵ,

$$\frac{\mathrm{d}\left(\frac{x}{2-\frac{\epsilon}{x}}-\epsilon\right)}{\mathrm{d}\,\epsilon}=\frac{x^2}{(\epsilon-2x)^2}-1<0,$$

where the inequality holds since $\epsilon < x$. (Note that $\delta < 0$ implies that $\epsilon < x$) The proof for step (ii) is identical to the proof for its counterpart in the case of $\delta > 0$.

G Theorem 6.7

In this section, we provide a proof sketch for Theorem 6.7. The proof needs to consider multiple cases, depending on the set of the customers served by the monopolist, demonstrated by Figure 14. In this proof sketch, we are going to consider Case 1 only. In fact, one can show that Case 3 never happens (the monopolist always servers less than half of the customers) and therefore, the complete proof considers only one additional case, Case 2. In here, we present the proof for Case 1; the proof for Case 2 is similar.

Our approach involves defining another planner, whom we call the double-monopolist, whose goal is maximizing profit by posting a price and wage (p, w). The difference between the double-monopolist and the monopolist is that the former owns two firms (as defined previously in a duopoly) but uses the same payment profile at both firms, whereas the latter owns only one firm while facing the same customers and workers as the double-monopolist. The optimal solution to the double-monopolist's problem is called the double-monopoly equilibrium.

For any planner $\mathsf{pl} \in \{\mathsf{mon}, \mathsf{dm}\}$, the optimal solution to the planner's problem is denoted by $(p_{\mathsf{pl}}, w_{\mathsf{pl}}, k_{\mathsf{pl}})$. For example, the optimal solution to the double-monopolist's problem is defined by $(p_{\mathsf{dm}}, w_{\mathsf{dm}}, k_{\mathsf{dm}})$. For $\mathsf{pl} = \mathsf{duo}$, $p_{\mathsf{duo}}, w_{\mathsf{duo}}$ denote the price and wage at the symmetric duopoly equilibrium, while k_{duo} denotes the *total* rate of customers who request service in the duopoly equilibrium.

Consistent with our usual notation, we use the variable b to denote $p + c(mw - k)$. For example, $b_{\mathsf{mon}} = p_{\mathsf{mon}} + c(mw_{\mathsf{mon}} - k_{\mathsf{mon}})$. This quantity is also called the *aggregate cost* that customers incur. Let the function $D : \mathbb{R}_+ \to [0, 1]$ determine the customers' demand function, i.e. $D(b)$ is the rate of customers who request service when the aggregate cost that a customer incurs at the firm is b. Similarly, and with slight abuse of notation, we use $D(b_1, b_2)$ to denote the rate of customers who join firm 1 when the aggregate costs at firms 1,2 respectively are b_1, b_2. Recall that this function is defined in Subsection E.1.

The proof involves two steps. In Step 1, we show that $w_{\mathsf{mon}} < w_{\mathsf{dm}}$, and in Step 2 we will prove that $w_{\mathsf{dm}} < w_{\mathsf{duo}}$. We sketch the proof in each step below. A key fact that will be used in the proof is that the following equations all hold:

$$c'(mw_{\mathsf{mon}} - k_{\mathsf{mon}}) = -1/m, \tag{G.1}$$

$$c'(mw_{\mathsf{duo}} - k_{\mathsf{duo}}) = -1/m, \tag{G.2}$$

$$c'(mw_{\mathsf{dm}} - k_{\mathsf{dm}}) = -1/m. \tag{G.3}$$

We have proved the second equation in Subsection E.1. The proof for the first and third equations are similar to the proof for (B.6). We do not repeat the proofs here. Intuitively, these equations are saying that for a marginal increase of ϵ in the wage, the firm can increase price by ϵ without changing the rate of customers who join. In other words, the marginal rate of substitution between *price* and *minus wage* is equal to 1 on any iso-quant of the customer's payoff function, the function $u(v) = v - p - c(i)$ for a customer with valuation v.

Proof sketch for Step 1. The proof is by contradiction. Suppose $w_{\mathsf{mon}} > w_{\mathsf{dm}}$. Then, by (G.1) and (G.3), we must have $k_{\mathsf{mon}} > k_{\mathsf{dm}}$. For each planner, namely $\mathsf{pl} \in \{\mathsf{mon}, \mathsf{dm}\}$, define the *adjusted price elasticity*[26] of customers' demand for that planner as

$$\mathfrak{E}_{\mathsf{pl}} = -\frac{k'_{\mathsf{pl}}(p_{\mathsf{pl}})}{k_{\mathsf{pl}}(p_{\mathsf{pl}})} \cdot (p_{\mathsf{pl}} - w_{\mathsf{pl}}) = -\frac{D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}})) \cdot b'(p_{\mathsf{pl}})}{D_{\mathsf{pl}}(b_{\mathsf{pl}}(p_{\mathsf{pl}}))} \cdot (p_{\mathsf{pl}} - w_{\mathsf{pl}}), \tag{G.4}$$

where the parameters in the above equation are defined as follows. For any planner pl, the function $k_{\mathsf{pl}}(p)$ denotes the rate of customers who join the firm as a function of the price posted by the firm, p. (Wage is held fixed, its value being equal to its equilibrium value,

[26]Recall the definition of adjusted price elasticity from Subsection 6.2, where we also explain the rationale behind this terminology.

w_{pl}.) Similarly, $b_{\mathsf{pl}}(p)$ denotes the aggregate cost that customers face as a function of p. The function $D_{\mathsf{pl}}(\cdot)$ is defined as before. Note that all these functions depend on the value of pl, and so it appears as a subscript.

The FOC for price implies that $\mathfrak{L}_{\mathsf{pl}} = 1$ should hold for all planners. To complete the proof in this step, we will show that $\mathfrak{L}_{\mathsf{dm}} > 1$, which will be a contradiction. (Intuitively, this would mean that planner dm can increase profit by reducing the price, which would be a contradiction.)

A key step is observing that

$$\begin{aligned} k'_{\mathsf{pl}}(p_{\mathsf{pl}}) &= D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}})) \cdot b'(p_{\mathsf{pl}}) \\ &= D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}})) \cdot \frac{\mathrm{d}(p_{\mathsf{pl}} + c(mw_{\mathsf{pl}} - k_{\mathsf{pl}}))}{\mathrm{d}\,p} \\ &= D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}})) \cdot \left(1 + \frac{-1}{m} \cdot (-k'_{\mathsf{pl}}(p_{\mathsf{pl}}))\right), \end{aligned} \tag{G.5}$$

where (G.5) holds by (G.1),(G.2), and (G.3). Solving for $k'_{\mathsf{pl}}(p_{\mathsf{pl}})$ implies that

$$k'_{\mathsf{pl}}(p_{\mathsf{pl}}) = \frac{D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}))}{1 - D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}))/m}. \tag{G.6}$$

We now use (G.6) to rewrite (G.4) as

$$\mathfrak{L}_{\mathsf{pl}} = \left(-\frac{D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}))}{D_{\mathsf{pl}}(b_{\mathsf{pl}}(p_{\mathsf{pl}}))} \cdot \frac{1}{1 - D_{\mathsf{pl}}{}'(b_{\mathsf{pl}}(p_{\mathsf{pl}}))/m}\right) \cdot (p_{\mathsf{pl}} - w_{\mathsf{pl}}). \tag{G.7}$$

Let us denote the first and second multiplicands on the RHS by f_{pl} and g_{pl}, respectively. We will show that $f_{\mathsf{dm}} \geq f_{\mathsf{mon}}$ and $g_{\mathsf{dm}} > g_{\mathsf{mon}}$, which will imply that $\mathfrak{L}_{\mathsf{dm}} > \mathfrak{L}_{\mathsf{mon}}$, which is a contradiction. Proving $g_{\mathsf{dm}} > g_{\mathsf{mon}}$ is easy: we know that $w_{\mathsf{dm}} < w_{\mathsf{mon}}$ holds by assumption. On the other hand, because $k_{\mathsf{mon}} > k_{\mathsf{dm}}$ holds, then we should also have $b_{\mathsf{dm}} > b_{\mathsf{mon}}$. (G.1) and (G.3) then imply that $p_{\mathsf{dm}} > p_{\mathsf{mon}}$. Therefore, $g_{\mathsf{dm}} > g_{\mathsf{mon}}$.

To complete Step 1, it remains to show that $f_{\mathsf{dm}} \geq f_{\mathsf{mon}}$. To this end, we first do a slight abuse of notation and write f_{dm} as a function of b (i.e. $f_{\mathsf{dm}}(b)$, so that $f_{\mathsf{dm}}(b_{\mathsf{dm}}) = f_{\mathsf{dm}}$. Note that the LHS of the equality denotes the value of the function evaluated at $b = b_{\mathsf{dm}}$, and the RHS is a scalar denoting the value of f_{dm} which was defined before). First, we show that $f_{\mathsf{dm}}(b)$ is an increasing function of b. This is done in file "wage-thm-case1-a". Then, we find a b^* such $D_{\mathsf{dm}}(b^*) = D_{\mathsf{mon}}(b_{\mathsf{mon}})$. By monotonicity of the functions $f_{\mathsf{dm}}(b)$ and $D_{\mathsf{dm}}(b)$ and the fact that $b_{\mathsf{dm}} > b_{\mathsf{mon}}$, to prove the claim that $f_{\mathsf{dm}} \geq f_{\mathsf{mon}}$, it suffices to prove $f_{\mathsf{dm}}(b^*) \geq f_{\mathsf{mon}}$.

This is what we show in files "wage-thm-case1-b" and "wage-thm-case1-c".

Proof sketch for Step 2. In this step, we prove that $w_{dm} < w_{duo}$. Note that by (G.2) and (G.3), we have

$$w_{dm} < w_{duo} \Leftrightarrow k_{dm} < k_{duo}.$$

The proof is by contradiction. Suppose $w_{dm} > w_{duo}$, which also means $k_{dm} > k_{duo}$. Because the former fact implies $b_{dm} < b_{duo}$, and also because of (G.2) and (G.3), $p_{dm} < p_{duo}$ must hold as well.

The rest of the argument in this step is summarized as follows. A standard payment profile for the double-monopolist is a profile that satisfies (G.3). Similarly, a standard payment profile for a firm in a duopoly is a profile that satisfies (G.2). In the first part of the argument, we will show that the adjusted price elasticity of the customers' demand for double-monopolist computed at a standard payment profile with aggregate cost b is increasing in b. In the second part, we will show that the payment profile used in the duopoly equilibrium is a standard payment profile for the double-monopolist, and that at this payment profile, the adjusted price elasticity of customers' demand for the double-monopolist is larger than the adjusted price elasticity of the customers' demand for a firm in the duopoly equilibrium (which is just equal to 1). This will be a contradiction, as the adjusted price elasticity of customers' demand for the double-monopolist computed at the double-monopoly equilibrium is equal to 1.

To formalize the argument, we first write the adjusted price elasticity of the customers' demand for the double-monopolist at a standard payment profile as a function of the customers' aggregate cost, b. For this, we use the notation $(p_{dm}(b), w_{dm}(b))$ to denote the double-monopolist's standard payment profile as a function of b. Recall the definition of adjusted price elasticity from (G.4). Evaluated at the standard payment profile with aggregate cost b, the adjusted price elasticity is equal to

$$\mathfrak{E}_{dm}(b) = -\frac{k'_{dm}(p_{dm}(b))}{k_{dm}(p_{dm}(b))} \cdot (p_{dm}(b) - w_{dm}(b)) = -\frac{D_{dm}'(b) \cdot b'(p_{dm}(b))}{D_{dm}(b)} \cdot (p_{dm}(b) - w_{dm}(b)). \quad \text{(G.8)}$$

Now, because we are working with a standard payment profile, we can use (G.7) to rewrite (G.8) as follows:

$$\mathfrak{E}_{dm}(b) = \left(-\frac{D_{dm}'(b)}{D_{dm}(b)} \cdot \frac{1}{1 - D_{dm}'(b)/m}\right) \cdot (p_{dm}(b) - w_{dm}(b)). \quad \text{(G.9)}$$

There are two multiplicands on the RHS of (G.9). We denote these multiplicands as functions of b, with the first and second multiplicand denoted as $f(b), g(b)$, respectively. In Step 1 we showed that $g(b)$ is increasing in b. To complete the proof in this step, we just need to show that $f(b)$ is increasing in b. This would prove that $\mathfrak{E}_{\mathsf{dm}}(b)$ is increasing in b.

In files "wage-thm-db-monotone-el-case1" and "wage-thm-db-monotone-el-case2", we prove that $f(b)$ is increasing in b. These files correspond to separate cases concerning b, which are defined in Figure 15. In the first and second file we consider Cases 1 and 2, respectively. In each file we derive a closed-form expression for $\mathfrak{E}_{\mathsf{dm}}(b)$.

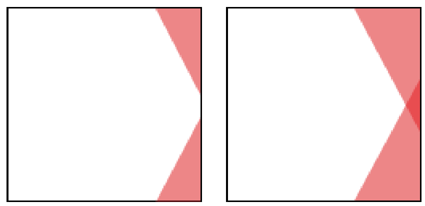


Figure 15: From left to right: Case 1 and Case 2.

In the last part of the argument, we show that at any standard payment profile with aggregate cost b, the adjusted price elasticity of the customers' demand for the double-monopolist is larger than the adjusted price elasticity of the customers' demand for a firm in the duopoly equilibrium. This will show that the promised contradiction holds.

First, note that for a fixed b, the standard payment profiles of the double monopolist and a firm in the duopoly equilibrium should be identical, because of (G.2) and (G.3). Now, we use (G.6) to compare the two price-elasticities, while noting the the second multiplicand on the RHS (the term $p_{\mathsf{pl}} - w_{\mathsf{pl}}$) is equal for both planners. So, it remains to show that the first multiplicand on the RHS is smaller for the double-monopolist. This is merely an algebraic calculation, and is done in files "wage-thm-db-el1" and "wage-thm-db-el2". These two files prove the claim for two separate cases concerning b: Cases 1 and 2 defined in Figure 15.